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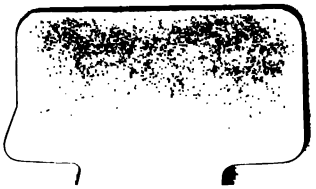
KEY TO
HUGHES'S
MENSURATION



Hughes's Educational Course.

KEY TO MENSURATION.

FOR PUPIL TEACHERS.

By JAMES ELLIOT,

PROFESSOR OF MATHEMATICS IN QUEEN'S COLLEGE, LIVERPOOL;
AUTHOR OF 'A COMPLETE TREATISE ON PRACTICAL GEOMETRY AND MENSURATION,'
'A COMPLETE TREATISE ON THE NATURE AND USE OF LOGARITHMS,
AND ON PLANE TRIGONOMETRY,' ETC.

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CHAPTER III.

MENSURATION OF LINES.

PROBLEM I.

ANSWERS TO UNRESOLVED EXERCISES.

A. 65.	D. 2·828+.	G. 44 feet.
B. 195.	E. 1·414+.	H. 120 yards.
C. 6 ft. 8 in.	F. 8·385+.	K. 16 feet from it.

OPERATION OF EXERCISE 1.

$518^2 = 263{,}169$
$684^2 = 467{,}856$
$\text{Hyp.}^2 = 731{,}025.$

EXERCISE B.

$445^2 = 198025$
$400^2 = 160000$
$\text{Perp.}^2 = 38025$

EXERCISE 2.

Reducing both dimensions to inches, we have—$\text{Perp.}^2 = 84^2 - 80^2 = 1156 - 900 = 256.$

EXERCISE 3.

$\text{Side}^2 = 95{\cdot}531076$
2
$\text{Diag.}^2 = 191{\cdot}062152$

Exercise 4.

$$\text{Diagonal}^2 = 101^2 \doteq 10201.$$
$$\text{Side}^2 \doteq 10201 \div 2 = 5100{\cdot}5.$$

Exercise 5.

$$\text{Perp.}^2 = 67{\cdot}5^2 \doteq 4556{\cdot}25$$
$$\text{Base}^2 = 36^2 \doteq 1296{\cdot}00$$
$$\text{Hyp.}^2 \doteq 5852{\cdot}25.$$

Exercise 6.

When we draw the diagonal, the field is, of course, divided into two $\triangle$s, in either of which the hypothenuse is the diagonal of the field, the base being one of the sides of the field, and the perpendicular the other.

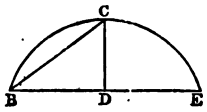
$$\text{Hyp.}^2 = 10{\cdot}71^2 \doteq 114{\cdot}7041$$
$$\text{Base}^2 = 9{\cdot}45^2 \doteq 89{\cdot}3025$$
$$\text{Perpendicular}^2 \doteq 25{\cdot}4016.$$

Exercise 7.

A perp. from the vertex to the base divides the isosceles Δ into two $\triangle$s, in either of which we have the hyp. = 65 and the base = 25. From these data we find the perp. = 60.

Exercise 8.

In the $\triangle$ BDC we have given BD = 5, CD = 4. Hence, by the rule, we find $BC = \sqrt{41}$.



Exercise 9.

In this question we have given BC = 11·8 and BD = 8·25. CD is required.

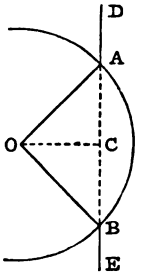
$$BC^2 \doteq 127{\cdot}6900$$
$$BD^2 \doteq 68{\cdot}0625$$
$$CD^2 \doteq 59{\cdot}6275.$$

Exercise F.

We have a $\triangle$, of which one leg is $7\frac{1}{2}$, and the other $3\frac{3}{4}$. Hence,

$$\text{Hyp.} \doteq \sqrt{(7{\cdot}5^2 + 3{\cdot}75^2)} = \sqrt{(56{\cdot}25 + 14{\cdot}0625)}$$
$$= \sqrt{(70{\cdot}3125)} = 8{\cdot}385+.$$

Exercise 10.



Let O be the centre of the axle; OA or OB, the shaft; and DE, the wall: then AB is the extent of wall which must be taken down.

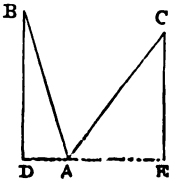
Now in the $\triangle$ ACO, AO $\doteq 15$, and OC $\doteq 10\frac{1}{2}$. $\therefore$, by the rule, AC $\doteq 10{\cdot}71+$, and AB = twice AC $\doteq 21{\cdot}4+$.

Exercise G.

Let AB and AC be the ladder in its two positions, and DE the street. Then,

In $\triangle$ ADB,	In $\triangle$ AEC
AB $\doteq 50$,	AC $\doteq 50$,
BD $\doteq 48$.	CE $\doteq 40$.
$\therefore$ AD $\doteq 14$.	$\therefore$ AE $\doteq 30$.

$\therefore$ DE $\doteq 14 + 30 \doteq 44$.



Exercise 11.

Here $h \doteq 100$, and $p \doteq 99{\cdot}5$

$\therefore h^2 \doteq 10000$

and $p^2 \doteq 9900{\cdot}25$

$\therefore b^2 \doteq 99{\cdot}75$.

Exercise 12.

If a perp. be drawn from the vertex to the base, it divides the equil. $\triangle$ into two $\triangle$s, in each of which, $h \doteq 1$ and $b \doteq \cdot 5$, giving $p^2 = \cdot 75$.

Exercise H.

The footpath divides the field into two $\triangle$s, in either of which the legs are 216 and 195. Hence we find the hypothenuse 291. The sum of the two legs is $216 + 195 = 411$. Therefore the distance saved is $411 - 291 = 120$.

Exercise K.

As the ladder is *first* placed we have a $\triangle$, of which the hyp. is 65, and the base 33, and, consequently, the perp. 56. Therefore the height of the wall is $56 + 7 = 63$.

In the *proposed* position of the ladder the hyp. is again 65; but the perp. is 63. From these data we find the base 16.

Problem VI.

Note. Archimedes, the famous Mathematician of Syracuse, was the first to compute the ratio of the diameter of a circle to the circumference with any considerable degree of accuracy. He proves that it must be the ratio of 1 to a number somewhere between $3\frac{1}{7}$ and $3\frac{10}{71}$.* The former of these, or the ratio of 7 to 22, forms a convenient practical rule, and when expressed decimally is correct to *three* figures, or two decimal places. In the fifth century of our era, Sporus extended the calculation as far as 10,000th parts, which is equivalent, decimally, to *four or five* figures. The introduction of the Arabic or modern notation into Arithmetic, much facilitated this, as well as all other calculations. In the end of the sixteenth century the number π was com-

* The method employed for this purpose by Archimedes and those who followed him, till the discovery of Fluxions, will be shown in this volume, *Chapter* IV. *Pr.* XIII.

puted by Vieta, a French Mathematician, as far as *eleven* figures; and, soon after this, it was simplified in its form of expression, and still farther extended, by Metius, Adrianus Romanus, and Van Ceulen, all natives of the low countries, —the first reducing it to the very simple fractional value, $\frac{355}{113}$, which, expressed decimally, is correct to *seven* figures, —the second, Adrianus, carrying the decimal expression to *seventeen* figures, and Van Ceulen to *thirty-six*, namely, 3·141,592; 653,589; 793,238; 462,643; 383,279; 502,88+. The last was regarded as so wonderful an attainment that the number was engraved on his tombstone. This achievement of unwearied industry could not then be surpassed, and the number, though verified by Snellius (also a Dutchman), was never extended farther till Sir Isaac Newton's discovery of Fluxions opened the way to methods so much more expeditious, that Van Ceulen's result, which cost two years of constant toil, can now be attained in a few hours. It was subsequently extended by Machin, in England, to 100 figures, by De Lagney and others, in France, still farther, and recently, by M. Dase to 200, Dr Clausen to 250, Dr Rutherford to 440, and Mr Shanks to 530 figures.

METHODS OF CALCULATING THE NUMBER π.

I. We know that the circumference is less than that of any polygon circumscribed about it, and greater than that of any polygon inscribed in it. We may, therefore, describe a square in a circle whose diameter is 1, and another square about the same circle, the circumferences of which squares will be 2·8+ and 4·0. The circumference of the circle must be somewhere between these; but, as they differ so far, they are of no use. We therefore find, next, the circumferences of two octagons described in the same manner, which are 3·06+ and 3·31+. As these agree only in the first figure, we have, as yet, found no more than one figure correct. We therefore proceed in the same way with polygons of 16, 32, 64, &c. sides. Or, instead of the square, we may commence with the hexagon, and proceed from it to polygons of 12, 24, 48, &c. sides.* This method, how-

* *A very neat mode of doing this will be found in Hutton's Course of Mathematics, Mensuration of Planes*, Pr. VII

ever (whether we start with the square or the hexagon), is very tedious. We therefore go on to the others.

II. We find the area of the circle in the manner explained in this volume, Ch. IV., Pr. XIII., and divide the area thus found by half the radius. This method is also very laborious.

III. By the method of *Fluxions*, or, in other words, by the Differential and Integral Calculus, thus :—

Let z be any arc of a circle and x its tangent, the radius being 1. Then, as may be found in any treatise on the Differential Calculus,

$$\partial z \doteq \frac{\partial x}{1+x^2} = \partial x \times \frac{1}{1+x^2}.$$

$$\text{But } \frac{1}{1+x^2} \doteq 1 - x^2 + x^4 - x^6 + \&c.$$

$$\therefore \partial z \doteq \partial x - x^2\,\partial x + x^4\,\partial x - x^6\,\partial x + \&c.$$

By integrating the successive terms of this series, we obtain (the constant being = 0).

$$z = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \&c.$$

Having found this *general* formula, we must, in order to apply it to our purpose, give a *particular* value to x and z. Let us take for z an arc of 45°, or $\frac{1}{8}$ of the circumference. Then $x = \tan 45° \doteq 1$.

$$\therefore \text{Arc of } 45° \doteq 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} + \&c.$$

Having calculated the sum of this series to any extent required, we multiply that sum by 4, for the semicircumf. to the radius 1, which is equal to the whole circumf. to the diameter 1, or the number Π. The series is very simple and very easily calculated; but it converges so slowly, that, to obtain a result correct only to a few figures, the number of terms required is immense, and the labour very great.

To obtain a more rapidly converging series, let us take an arc of 30°, the tangent of which is $1 \div \sqrt{3}$, and let us, at the same time, make a slight change in the mode of expressing the general formula, thus—

$$z = x(1 - \tfrac{1}{3}x^2 + \tfrac{1}{5}x^4 - \tfrac{1}{7}x^6 + \&c.)$$

When the arc is of 30°, x being then the tangent of 30°, is easily found to be $1 \div \sqrt{3}$; for $\sin 30° \doteq \frac{1}{2}$, $\cos 30° = \sqrt{\frac{3}{4}} = \frac{1}{2}\sqrt{3}$, *and* cos : sin :: rad : tan, or $\frac{1}{2}\sqrt{3} : \frac{1}{2} :: 1 : 1 \div \sqrt{3}$.

$$\therefore x \doteq \frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3},\ x^2 = \frac{1}{3},\ x^3 = \frac{1}{3^2},\ x^6 = \frac{1}{3^3},\ \&c.$$

$$\therefore \text{Arc of } 30° \doteq \frac{1}{3}\sqrt{3}\left(1 - \frac{1}{3\cdot 3^1} + \frac{1}{5\cdot 3^2} - \frac{1}{7\cdot 3^3} + \frac{1}{9\cdot 3^4} - \&c.\right),$$

and semicircumf. to rad. 1 = circumf. to diam. 1 =

$$\Pi \doteq 2\sqrt{3}\left(1 - \frac{1}{3\cdot 3^1} + \frac{1}{5\cdot 3^2} - \frac{1}{7\cdot 3^3} + \frac{1}{9\cdot 3^4} - \&c.\right)$$

The last formula, first proposed by Halley, was used by Abraham Sharp in calculating the circumference to 72 places of figures, by Machin in calculating it to 100, and by De Lagney, to 128.

The actual process of computation is as follows:—

Putting Q to represent the quotient of $1 \div 3$, or the fraction $\frac{1}{3}$, Halley's formula becomes

$$\Pi = 2\sqrt{3} \times (1 - \tfrac{1}{3}Q + \tfrac{1}{5}Q^2 - \tfrac{1}{7}Q^3 + \tfrac{1}{9}Q^4 - \&c.)$$

We therefore, first of all, find the values of Q, Q^2, Q^3, &c. each of which we carry to seven decimal places, that we may have the result true to six figures—the extent to which we now propose to carry our calculation.

3)1·000,000,0	3)·333,333,3
3) ·333 333 3 = Q	·111 111 1 = $\frac{1}{3}Q$.
3) ·111 111 1 = Q^2	5)·111 111 1
3) ·037 037 0 = Q^3	*22 222 2 = $\frac{1}{5}Q^2$.
3) ·012 345 7 = Q^4	7) 37 037 0
3) ·004 115 2 = Q^5	5 291 0 = $\frac{1}{7}Q^3$.
3) ·001 371 7 = Q^6	9) 12 345 7
3) ·000 457 2 = Q^7	1 371 7 = $\frac{1}{9}Q^4$.
3) ·000 152 4 = Q^8	11) 4 115 2
3) ·000 050 8 = Q^9	374 1 = $\frac{1}{11}Q^5$.
3) ·000 016 9 = Q^{10}	13) 1 371 7
3) ·000 005 6 = Q^{11}	105 5 = $\frac{1}{13}Q^7$.
3) ·000 001 9 = Q^{12}	15) 457 2
·000 000 6 = Q^{13}	30 5 = $\frac{1}{15}Q^7$.

17)152,4	19)50,8	
9 0 = $\frac{1}{17}Q^8$	2 7 = $\frac{1}{19}Q^9$	
21)16,9	23)5,6	25)1,9
8 = $\frac{1}{21}Q^{10}$	2 = $\frac{1}{23}Q^{11}$	1 = $\frac{1}{25}Q^{12}$

* *It is not thought necessary any further to insert the point and the ciphers, as all the lines terminate together.*

We now proceed to collect the terms already computed.

Positive Terms.		*Negative Terms.*	
1	$= 1{\cdot}000{,}000{,}0$	$\frac{1}{3}\,Q$	$= {\cdot}111{,}111{,}1$
$\frac{1}{5}\,Q^2$	$= 22\,222\,2$	$\frac{1}{7}\,Q^3$	$= 5\,291\,0$
$\frac{1}{9}\,Q^4$	$= 1\,371\,7$	$\frac{1}{11}\,Q^5$	$= 374\,1$
$\frac{1}{13}\,Q^6$	$= 105\,5$	$\frac{1}{15}\,Q^7$	$= 30\,5$
$\frac{1}{17}\,Q^8$	$= 9\,0$	$\frac{1}{19}\,Q^9$	$= 2\,7$
$\frac{1}{21}\,Q^{10}$	$= 8$	$\frac{1}{23}\,Q^{11}$	$= 2$
$\frac{1}{25}\,Q^{12}$	$= 1$	$\frac{1}{27}\,Q^{13}$	$= 0$
Sum $=$	$+1{\cdot}023\,709\,3$	Sum $=$	${\cdot}116\,809\,6.$
	$-0{\cdot}116\,809\,6$		
	${\cdot}906\,899\,7$		
	$8\,050\,237{\cdot}1 = \sqrt{3}$, reversed.		
	$906\,899\,7$		
	$634\,829\,8$		
	$27\,207\,0$		By contracted multiplication.
	$1\,813\,8$		
	$45\,3$		
	7		
	$1{\cdot}570\,796\,3$		
	2		
	$3{\cdot}141\,592\,6 = \Pi.$		

The result we have obtained is carried to eight figures; but as the sum of the series was carried only to seven figures, we cannot regard more than six as certain, the seventh being only probably correct, and the eighth, though right, a mere chance figure.

To carry the result twice as far, would require about four times the labour; to carry it three times as far, would require about nine times as much; and to obtain the result correct to 36 figures (Van Ceulen's extent), would call for an amount of figures and of labour about 36 times as great as the work exhibited above: 72 figures (Sharp's extent), would involve 144 times as much, and 128 figures (De Lagney's limit) could not be done with less than 450 times.

After the great extent to which the expression was carried by the calculators already mentioned, any simplification of the mode could only be a subject of curiosity, as there is no probability that the result can ever be required *to any* extent even approaching that which they

attained. After Machin had finished his task, however, he showed how it might have been effected much more easily by a peculiar mode of applying the general formula, and other authors have shown similar modes. Thus Euler pointed out that the arc whose tangent is $\frac{1}{2}$, and the arc whose tangent is $\frac{1}{3}$, together make an arc of 45°. If therefore the length of these two arcs be computed and added together, and the sum be multiplied by 4, we shall then have the number π, and the two series will both converge more quickly than that for the arc of 30°. Machin proposed to compute the arc whose tangent is $\frac{1}{5}$, and then the arc whose tangent is $\frac{1}{239}$, and showed that the latter of these, taken from four times the former, would make an arc of 45°.

Dr Hutton, in his *Tracts*, has shown a general method of which Euler's and Machin's are only particular cases. That method will be found fully detailed in Tract 18 of Vol. 1, and also in the Key to the Author's "Complete Treatise," page 22 to 25, in which the best of Hutton's particular methods is considerably improved for use and exhibited in actual calculation.

The process which Dr Rutherford at first adopted is still more concise than any of them, when the calculation is required to be carried to any great extent, and when the easy mode of dividing by 99 is employed. He proceeded upon a formula constructed on the same principle as Euler's, which is, that an arc of 45° $\doteq$ 4 times an arc whose tangent is $\frac{1}{5}$ − an arc whose tangent is $\frac{1}{70}$ + an arc whose tangent is $\frac{1}{99}$. Dr Rutherford subsequently returned to Machin's formula, which was also that used by Mr Shanks. For more full details of Dr Rutherford's first method, see the Philosophical Transactions for 1841, and the new Course of Mathematics for the use of the Royal Military Academy.

ANSWERS TO UNRESOLVED EXERCISES.

A. 81·68+.	C. 46 ft. 1 − in.	E. 11·727−.
B. 58·905−.	D. 0·6366+.	F. 2·997+.

Exercise 4.

As the whole circumf. $\doteq$ 3·1416 times the diameter, so the half circumf. $\doteq$ 3·1416 times the radius.

Exercise 5.

Radius $\doteq 29\frac{1}{2}$. $\therefore$ Diam. $\doteq 59$ inches.
Hence we find circumf. $= 185{\cdot}3544$.

Exercise 6.

For this we use the reverse formula, $D = C \div \pi$. That is, we divide $6{\cdot}224$ by $3{\cdot}1416$. If we use the second rev. formula, we multiply $6{\cdot}224$ by $\cdot 31831$.

Exercise 7.

A mile being 5280 feet, the thousandth part of that, or $5{\cdot}28$ feet, will be the circumf. of the wheel. This we divide by $3{\cdot}1416$, or multiply by $\cdot 31831$.

Exercise F.

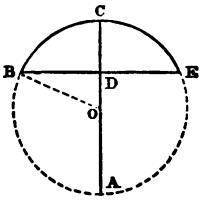
$9\frac{5}{12} \doteq 9{\cdot}41\dot{6}$.

Problem VII.

Answers to unresolved exercises.

A. $6{\cdot}34375$. | B. $42{\cdot}5$. | C. $475{\cdot}9+$.

Exercise 1.



$CD \doteq 1{\cdot}5$; $BD \doteq 5{\cdot}4$; $BD^2 \doteq 29{\cdot}16$.
$\therefore AD \doteq 29{\cdot}16 \div 1{\cdot}5 = 19{\cdot}44$.
$AC \doteq 20{\cdot}94$, and $OC = 10{\cdot}47$.

Exercise 2.

$CD \doteq \frac{21}{4}$; $BE \doteq \frac{21}{4}$; and $BD \doteq \frac{21}{8}$.
$\therefore AD \doteq (\frac{21}{8})^2 \div \frac{21}{4} = \frac{21}{8} \times \frac{21}{8} \times \frac{4}{21} = \frac{21}{16} = 1{\cdot}3125$.
$AC \doteq 5{\cdot}25 + 1{\cdot}3125 = 6{\cdot}5625$.

Exercise 3.

In $\triangle$ BDO we have BO = 100, and BD = 28.
$\therefore$, by Pr. I., OD $\doteq$ 96.
CD $\doteq$ 100 − 96 = 4.

Exercise 4.

Here CD $\doteq$ 0·7, and OC = 1. $\therefore$ OD $\doteq$ 0·3.
Then, in $\triangle$ BDO, BO $\doteq$ 1, and OD = 0·3.
$\therefore$ $OD^2 \doteq 0{\cdot}09$; and $BD^2 \doteq 0{\cdot}91$.
$\therefore$ BD $\doteq$ 0·9539 +, and BE = 1·908 −.

Exercise A.

$$AD \doteq \left(\frac{6{\cdot}3}{2}\right)^2 \div 2{\cdot}8 = \frac{6{\cdot}3 \times 6{\cdot}3}{4 \times 2{\cdot}8} = \frac{6{\cdot}3 \times 0{\cdot}9}{4 \times 0{\cdot}4} = 3{\cdot}54375.$$

To this add CD = 2·8.

Exercise B.

$$\text{Diam.} \doteq \frac{(17 \times 2)^2}{17} + 17 = 68 + 17 = 85.$$

Problem VIII.

Answers to unresolved exercises.

A. 23·4.
B. $15\frac{1}{2}$.
C. 44·72 +.
D. 1·06̇.

Exercise 1.

BC $\doteq$ 15 ft. 7 in. = 187 in.
CD $\doteq$ 7 ft. $9\frac{1}{2}$ in. = 93·5 in.
$\therefore$ $AC \doteq \frac{187 \times 187}{93{\cdot}5} = 187 \times 2.$
$\therefore$ OC $\doteq$ 187 in. = 15 ft. 7 in.

Exercise 2.

Here, diam. AC $\doteq 9$, and CD $= 4$.
$\therefore$, by 2d rev. for., BC $\doteq \sqrt{(9 \times 4)} = \sqrt{36}$.

Exercise B.

$$\text{Diam.} \doteq (15\tfrac{1}{2})^2 \div 7\tfrac{3}{4} = \tfrac{31}{2} \times \tfrac{31}{2} \times \tfrac{4}{31} = 31.$$

Exercise C.

By the second reverse formula,

$$\text{BC} \doteq \sqrt{(200 \times 10)} = \sqrt{2000} = 44{\cdot}72+.$$

Exercise D.

$$\text{By first rev. for., CD} \doteq \frac{2{\cdot}4 \times 2{\cdot}4}{5{\cdot}4} = \frac{{\cdot}8 \times {\cdot}4}{{\cdot}3}.$$

Problem IX.

Answers to unresolved exercises.

A. 38. | B. 23·25 −.

Exercise 1.

As 360° : 57° 38′ :: 68 yards, or
As 21600′ : 3458′ :: 68 yds.; or, cancelling,
As 27 : 17·29 :: 17 yds. : 10 yds. 2·66 − ft.

Exercise 2.

As 360° : 1° :: 24,896 miles : 69·1$\dot{5}$ miles.

Problem X.

ANSWERS TO UNRESOLVED EXERCISES.

A. 1·658+. | B. 6·463+.

Exercise 1.

As 360° : 36° 42′ 16″ :: 25·1328; or
As 1296000″ : 132136″ :: 25·1328; or, cancelling,
As 27 : 16·517 :: 4·1888 : 2·5625−.

Exercise 2.

As 360° : 198° 7·3′ :: $25\frac{1}{3} \times 3{\cdot}1416$, or
As 21600 : 11887·3 :: 79·5872 : 43·800−.

Problem XII.

Note. The strict proof that O*b* : OB :: arc *bc* : arc BC (referring to the diagram in the *Course*) is as follows:—

If the whole circles were completed of which *bc* and BC are arcs, and if the circumferences of these two circles were named, respectively, 4*m* and 4*M*, *m* and *M* being their quadrants. Then the arcs *m* and *M* will be subtended by right angles at the centre (Euc. vi. 33).

2O*b* : 2OB :: 4*m* : 4*M*, (Playfair's Geom. Sup. i. 6).
∴ O*b* : OB :: *m* : *M*, (Euc. v. 15 and 11).
But *m* : arc *bc* :: ∟ : ∠O, (Euc. vi. 33),
and *M* : arc BC :: ∟ : ∠O, (Euc. vi. 33).
∴ *m* : arc *bc* :: *M* : arc BC,
and, alternately, *m* : *M* :: arc *bc* : arc BC.
But O*b* : OB :: *m* : *M*.
∴ O*b* : OB :: arc *bc* : arc BC.

The whole of this process is necessary unless we assume more than is to be found in our ordinary editions of Euclid, but the substance of the latter part of the demonstration is simply as given in the text of the *Course*.

ANSWERS TO UNRESOLVED EXERCISES.

A. 34·0(7+).	C. 4·65(7−).	E. 141(8).
B. 3·02(4−).	D. 63·7(0−).	F. 57·4(2+).

Exercise 1.

Nat. sine $\doteq 21 \div 24 =$ ·875.
Nearest on Table, ·8746.
Corresponding tabular arc, 1·0647.
$1{\cdot}0647 \times 24 \doteq 25{\cdot}5528$

Exercise 2.

Nat. versed-sine $\doteq 5 \div 10 =$ ·5.
Corresponding tabular arc, 1·0472.
$1{\cdot}0472 \times 10 \doteq 10{\cdot}472.$

Exercise A.

Nat. sine $\doteq 28 \div 32 =$ ·875. Hence as in Ex. 1, Arc $\doteq 1{\cdot}0647 \times 32.$

Exercise B.

N. v. s. $\doteq 1\frac{1}{2} \div 2\frac{3}{4} = \frac{3}{2} \times \frac{4}{11} = \frac{6}{11} =$ ·5455−.
Nearest on Table, ·5460.
Corresponding tabular arc, 1·0996.

Exercise 3.

Chord of whole arc $\doteq 600.$
Sine of half arc $\doteq 300.$ (See Note 3.)
Nat. sine of half arc $\doteq 300 \div 900 =$ ·3333 +.
Nearest on Table, ·3338.
Corresponding arc, ·3403.
$·3403 \times 900 \times 2 \doteq 612{\cdot}54.$

Exercise 4.

Nat. v. sine of half arc $\doteq 700 \div 725 = \cdot 9655$.
Nearest on Table, $\cdot 9651$.
Corresponding arc, $1\cdot 5359$.
$1\cdot 5359 \times 725 \times 2 \doteq 2227\cdot 055$.

Exercise C.

Nat. sine of half arc $\doteq 1\cdot 95 \div 2\cdot 3 = \cdot 8478+$.
Nearest on Table, $\cdot 8480$.
Corresponding arc, $1\cdot 0123$.

Exercise D.

Nat. v. s. of half arc $\doteq 12\cdot 81 \div 37\cdot 24 = \cdot 3440-$.
Nearest on Table, $\cdot 3439$.
Corresponding arc, $\cdot 8552$.

Exercise 5.

Here, by Pr. VII., diam. $\doteq 288 + 50 = 338$.
$\therefore$ radius $\doteq 169$.
N. sine of half arc $\doteq 120 \div 169 = \cdot 7101-$.
Nearest on Table, $\cdot 7092$.
Corresponding arc, $\cdot 7883$.
$\cdot 7883 \times 169 \times 2 \doteq 266\cdot 4454$.

Exercise 6.

Height, by Pr. I., $\doteq \sqrt{(25^2 - 20^2)} = 15$.
Diameter, by Pr. VIII., $\doteq 25^2 \div 15 = 1\frac{2}{3}\frac{5}{5}$.
Hence, radius $\doteq 1\frac{2}{6}\frac{5}{5}$. Therefore
N. sine of half arc $\doteq 20 \div 1\frac{2}{6}\frac{5}{5} = \frac{20}{1} \times \frac{6}{125} = \cdot 96$.
Nearest on Table, $\cdot 9596$.
Corresponding arc, $1\cdot 2857$.
$1\cdot 2857 \times 1\frac{2}{6}\frac{5}{5} \times \frac{2}{1} \doteq 53\cdot 57+$.

EXERCISE E.

We first find radius = 1250, as in Ex. 5, then
Nat. v. s. of half arc $\doteq 196 \div 1250 = \cdot1568$, or
Nat. sine of half arc $\doteq 672 \div 1250 = \cdot5376$.
Hence, nearest tabular arc $\doteq \cdot5672$.

EXERCISE 7.

Finding the supplemental arc, according to Note 2.

Nat. sine $\doteq 1\cdot8 \div 9 = \cdot2$.	
Nearest on Table,	$\cdot1994$.
Corresponding arc,	$\cdot2007$.
Sup. arc $\doteq \cdot2007 \times 9$	$= 1\cdot8063$.
Semicircumf. $\doteq 3\cdot1416 \times 9$	$= 28\cdot2744$.
Arc required,	$26\cdot4681$.

EXERCISE F.

Computing the supplemental arc, according to Note 2,

Nat. sine $\doteq 5\frac{1}{3} \div 20 = \cdot2667-$.	
Nearest on Table,	$\cdot2672$.
Corresponding arc,	$\cdot2705$.
Sup. arc $\doteq \cdot2705 \times 20$	$= 5\cdot4(10)$.
Semicircumf. $\doteq 3\cdot1416 \times 20$	$= 62\cdot832$.
Arc required,	$57\cdot4(22)$.

PROBLEM XIII.

DEMONSTRATION OF THE RULE.

Suppose we wish to find a rule for determining approximately the length of the arc from the chord of the arc and the chord of half the arc, putting a for the arc, we may, if we choose, assume the equation

$$a = Pc + Qc',$$

P and Q being co-efficients whose values are to be de-

termined. Then $c \doteq 2 \sin \frac{1}{2} a$, and $c' \doteq 2 \sin \frac{1}{4} a$, the radius being that of the required arc.

$$\therefore a = Pc + Qc' \doteq 2P \sin \tfrac{1}{2} a + 2Q \sin \tfrac{1}{4} a.$$

But m being any arc, and r its radius, we have, by a well-known formula,*

$$\text{Sin } m = m - \frac{1}{6} \cdot \frac{m^3}{r^2} + \frac{1}{120} \cdot \frac{m^5}{r^4} - \text{\&c.}$$

If we insert successively $\frac{1}{2} a$ and $\frac{1}{4} a$ in the place of m, we have

$$\text{Sin } \tfrac{1}{2} a = \frac{a}{2} - \frac{1}{6} \cdot \frac{a^3}{8r^2} + \frac{1}{120} \cdot \frac{a^5}{32r^4} - \text{\&c., and}$$

$$\text{Sin } \tfrac{1}{4} a = \frac{a}{4} - \frac{1}{6} \cdot \frac{a^3}{64r^2} + \frac{1}{120} \cdot \frac{a^5}{1024r^4} - \text{\&c.}$$

$$\therefore 2P \sin \tfrac{1}{2} a \doteq Pa - \frac{1}{6} \cdot \frac{P}{4} \cdot \frac{a^3}{r^2} + \frac{1}{120} \cdot \frac{P}{16} \cdot \frac{a^5}{r^4} - \text{\&c.}$$

$$\text{and } 2Q \sin \tfrac{1}{4} a = \frac{Q}{2} a - \frac{1}{6} \cdot \frac{Q}{32} \cdot \frac{a^3}{r^2} + \frac{1}{120} \cdot \frac{Q}{512} \cdot \frac{a^5}{r^4} - \text{\&c.}$$

$$\therefore a = 2P \sin \tfrac{1}{2} a + 2Q \sin \tfrac{1}{4} a \doteq$$

$$\left(P + \frac{Q}{2}\right) a - \frac{1}{6} \left(\frac{P}{4} + \frac{Q}{32}\right) \frac{a^3}{r^2} + \frac{1}{120} \left(\frac{P}{16} + \frac{Q}{512}\right) \frac{a^5}{r^4} - \text{\&c.}$$

If, then, we could find such values of P and Q that $\left(P + \frac{Q}{2}\right)$ should be $= 1$, and all the other co-efficients, $\left(\frac{P}{4} + \frac{Q}{32}\right)$, $\left(\frac{P}{16} + \frac{Q}{512}\right)$, &c. severally $= 0$, we should have an identical equation, viz. $a = a$, and a perfect rule instead of an approximation. But it is impossible to assign any such values to P and Q. All that we can do, is to find such values for them that we shall have

$$P + \frac{Q}{2} = 1 \text{.....(1)},$$

$$\text{and, } \frac{P}{4} + \frac{Q}{32} = 0 \text{.....(2)};$$

and we shall then have our formula holding true in the first and second terms of the series, and erring only in the

* *The demonstration of this formula may be found in the Key to the Author's "Complete Treatise," Part III., Pr. XL, or in any work on the higher Calculus.*

subsequent terms, which, we shall find, consist of small fractions, and are comparatively insignificant, especially if the arc is a small part of the circumference; that is, if a is small compared with r.

Resolving the two preceding equations to find the values of P and Q, we find $P = -\frac{1}{3}$ and $Q = +\frac{8}{3}$. Therefore returning to our assumed equation, viz. $a = Pc + Qc'$, and inserting for P and Q the values just found, we have

$$a = -\frac{1}{3}c + \frac{8}{3}c' = \frac{8c' - c}{3},$$

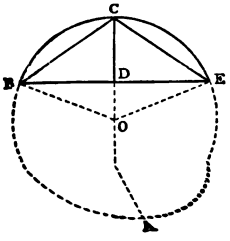
which, expressed in words, is our rule.

NOTE. To find a *general expression* for the *error* produced by this rule, see the Key to the Author's Complete Treatise on the same subject, Part III., Pr. XIII. It may be sufficient to say, here, that, when the arc is not of more than 60°, the error does not exceed the 6000th part of the whole, that, if the arc is short of 90° the error is less than 1 in 1300, and that if it comes up to 120°, the error is as much as 1 in 400. The deviation from the truth is insignificantly small for arcs of less than 30°, and so great as to render the rule entirely useless when the arc is nearly a semicircumference. The result, by this rule, always comes out *less* than it ought to be.

ANSWERS TO UNRESOLVED EXERCISES.

A. 34.	C. 12·0(5+).	E. 114·9−.
B. 6·48(1+).	D. 1419(−).	

EXERCISE 3.



Here we have BD = 24·25,
and CD = 18·25.
∴ BD² ≐ 588·0625,
and CD² ≐ 333·0625.
∴ BC² ≐ 921·1250,
and BC = 30·35+.

Exercise 4.

$$OD \doteq \sqrt{(BO^2 - BD^2)} = \sqrt{(10^2 - 8^2)} = 6.$$
$$CD \doteq 10 - 6 = 4.$$
$$BC \doteq \sqrt{(BD^2 + CD^2)} = \sqrt{(8^2 + 4^2)} = 8{\cdot}9443 -.$$

Exercise 5.

$$BC \doteq \sqrt{(CD \times CA)} = \sqrt{(4 \times 30)} = 10{\cdot}9545 -.$$
$$BD \doteq \sqrt{(CD \times DA)} = \sqrt{(4 \times 26)} = 10{\cdot}1980 +.$$
$$BE = 2BD \doteq 20{\cdot}3960 +.$$

Exercise B.

$$OD \doteq \sqrt{(BO^2 - BD^2)} = \sqrt{(3{\cdot}5^2 - 2{\cdot}8^2)} = 2{\cdot}1.$$
$$CD \doteq 3{\cdot}5 - 2{\cdot}1 = 1{\cdot}4.$$
$$BC = \sqrt{(BD^2 + CD^2)} \doteq \sqrt{(2{\cdot}8^2 + 1{\cdot}4^2)} = 3{\cdot}1305 -.$$

Exercise C.

$$BD^2 \doteq BO^2 - OD^2 = 5{\cdot}5^2 - 2{\cdot}5^2 = 24.$$
$$BC^2 \doteq BD^2 + DC^2 = 24 + 9 = 33.$$
$$\therefore BE = 2BD \doteq 2\sqrt{24} = 2 \times 4{\cdot}899 - = 9{\cdot}798 -,$$
$$\text{and } BC = \sqrt{33} = 5{\cdot}745 -.$$

Exercise D.

$$BC = \sqrt{(196^2 + 672^2)} = \sqrt{490000} = 700.$$

Exercise E.

The chord dividing the circle into two parts, and the greater arc being evidently a considerable part of the whole circumference, we find the smaller arc, by the rule, and subtract it from the whole circumference. The chord of half the smaller arc is $5{\cdot}38+$, and that arc itself is $10{\cdot}79+$. The circumf. of the whole circle is $125{\cdot}66+$; and the difference of these is $114{\cdot}87 \pm$.

Exercise 6.

From the great height of the arc compared with the length of the chord, we perceive that the arc is a large part of the whole circumf. We therefore proceed thus:—

$$\begin{aligned} \text{AD} &\doteq 60^2 \div 144 \text{ (by Pr. VII.)} = 25. \\ \text{Diam.} &\doteq 25 + 144 = 169. \\ \text{Chord BA} &\doteq \sqrt{(60^2 + 25^2)} = \sqrt{4225} = 65. \\ \text{Hence arc BAE} &\doteq 133{\cdot}33 + \\ \text{Circumf.} &= 169 \times 3{\cdot}1416 \doteq 530{\cdot}93 + \\ \therefore \text{Arc BCE} &\doteq 397{\cdot}60 \pm. \end{aligned}$$

Problem XIX.

DEMONSTRATION OF THE RULES.

The assumptions described in Notes 1 and 2, may, at first, have been merely hypothetical, verified by trials, but they may be made to rest on surer ground, and the amount of error may be ascertained in general terms, by deducing both rules from a true expression for the value of the elliptic circumference. That expression is the following:—

$$\Pi \times (a+b) \times (1 + \tfrac{1}{4}c^2 + \tfrac{1}{64}c^4 + \tfrac{1}{256}c^6 + \&c.)^*$$

a and b being the semiaxes, and c being the quotient of $(a-b) \div (a+b)$.

Since $(a-b)$ must always be less than $(a+b)$, c is always less than unity, and c^2, c^4, &c. constantly diminish as the powers advance. Rule 1, therefore, expressed by the formula, circumf. $= \Pi \times (a+b)$, will nearly agree with the true series when c is a small fraction, or when a is not much greater than b. The error will be expressed in general terms by

$$\Pi \times (a+b) \times (\tfrac{1}{4}c^2 + \tfrac{1}{64}c^4 + \tfrac{1}{256}c^6 + \&c.).$$

* This expression for the circumference of an ellipse was first given by Sir James Ivory in the Edinburgh Philosophical Transactions, Vol. IV. The demonstration may be found, expanded and simplified, in the Key to the Author's *Complete Treatise*, Part III., Problem XIX.

Rule 2, expressed by the formula, circumf. $= \Pi \times \sqrt{(2a^2 + 2b^2)}$, gives a closer approximation; for

$$2a^2 + 2b^2 \doteq (a+b)^2 + (a-b)^2 = (a+b)^2 + (a+b)^2 \times \left(\frac{a-b}{a+b}\right)^2$$
$$= (a+b)^2 + (a+b)^2 \times c^2 = (a+b)^2 \times (1+c^2).$$
$$\therefore \Pi \times \sqrt{(2a^2 + 2b^2)} \doteq \Pi(a+b) \times \sqrt{(1+c^2)}.$$
$$= \Pi(a+b) \times (1 + \tfrac{1}{2}c^2 - \tfrac{1}{8}c^4 + \tfrac{1}{16}c^6 - \&c.),$$

an expression which exceeds the true value given above by

$$\Pi \times (a+b) \times (\tfrac{1}{4}c^2 - \tfrac{9}{64}c^4 + \tfrac{15}{256}c^6 - \&c.).$$

Exercise 1.

By Rule 2. $\frac{1}{2}(3^2 + 2^2) \doteq \frac{1}{2}$ of $13 = 6{\cdot}5$.

$\sqrt{6{\cdot}5} \doteq 2\ 5495 +$.

$2{\cdot}5495 \times 3{\cdot}1416 \doteq 8{\cdot}0095 +$.

Exercise A.

By Rule 1, Circumf. $\doteq \dfrac{5+4}{2} \times 3{\cdot}1416 = 14{\cdot}1(4-)$.

By Rule 2, Circumf. $\doteq \sqrt{\dfrac{25+16}{2}} \times 3{\cdot}1416$

$= \sqrt{20{\cdot}5} \times 3{\cdot}1416 = 4{\cdot}5277 \times 3{\cdot}1416 = 14{\cdot}2(2+)$.

Problem XXVI.

Demonstration of the Rule.

The peculiar mode of demonstrating the rule for this problem will depend on the definitions given to the words "similar figures" and "corresponding lines, or lineal dimensions:" but, in all cases of *rectilineal* figures (or of *any* figures, when the two dimensions are both straight lines), the demonstration will be very simple if, in each figure, we connect the lines, which the two dimensions express, by one or more triangles. Then, (by Euc. v. 12) it may be shown that the same property applies not only to single straight lines, but to lines made up of any number *of corresponding* straight lines, such as the circumferences

of similar polygons. This being proved, the same property is next extended to curvilineal circumferences, by showing that it holds good in the case of the circumferences of similar polygons inscribed within them, whatever their number of sides, and however close their circumferences approach to those of the curvilineal circumferences. The demonstration is applied in the same way to curve lines which are *not complete* circumferences.

ANSWERS TO UNRESOLVED EXERCISES.

A. $7\frac{1}{2}$ and 5 inches.

B. The first; for the hat of the second ought to be $25\frac{7}{11}$ inches in circumf. to be proportional to that of the first.

C. 132·8 inches.	F. 7·778+.	K. 79·96 feet.
D. $3\frac{9}{11}$ nearly.	G. 4·243−.	L. 225 feet.
E. 6·675−.	H. 160·5+ft.	M. 260.

EXERCISE 1.

As 60 feet : 5 inches :: 25 feet : $2\frac{1}{12}$ inches.

EXERCISE 2.

As 5 inches : 60 feet :: 3·75 inches : 45 feet.

EXERCISE A.

As 300 feet : 12 inches :: 180 feet : $7\frac{1}{5}$ inches.
As 300 feet : 12 inches :: 125 feet : 5 inches.

EXERCISE 3.

As 72 inches : 142 inches :: $11\frac{1}{2}$ inches : $22\frac{4}{72}$ inches.

EXERCISE B.

As 68 inches : 72 inches :: 24 inches : $25\frac{3}{17}$ inches.

Exercise 4.

As 3·8 inches : 103 miles :: 11·44 inches : 310 + miles.

Exercise C.

As 5 miles : 332 miles :: 2 inches : 132·8 inches.

Exercise 5.

As 1000 : 136 in. :: 866 : $117\frac{3}{4}$ + in., or
As 1000 : 866 :: 11 ft. 4 in. : 9 ft. $9\frac{3}{4}$ in.

Exercise D.

As 22 : 12 :: 7 : $3\frac{9}{11}$.
Since 22 is not the exact number, neither will $3\frac{9}{11}$.

Exercise 6.

As 1·3066 : 1·2071 :: 6 : 5·543 +.

Exercise E.

As 68819 : 5·4 :: 85065 : 6·675 −.

Exercise 7.

As 1 mile : 600 miles :: $\frac{1}{10}$ in. : 15 inches.
$15 + 2 \times 2\frac{1}{2} = 20 +$.

Exercise 8.

As 126 feet : 84 feet :: 31 feet : 20 feet 8 in.

Exercise F.

As 1 : $5\frac{1}{2}$ :: 1·4142 : 7·778 + .

Exercise G.

The extreme length of every scale of chords is the chord of 90°, and the chord of 60° is the radius. The question therefore is simply this:—If the chord of 90° is 1·4142 when the radius is 1, what is the radius when the chord of 90° is 6 inches. We therefore say:

As 1·4142 : 6 :: 1 : 4·243 − .

Exercise 10.

As 9·86 feet : 132 feet :: 12 feet : 160·65 − feet.

Exercise H.

As 39·8 in. : 115·5 ft. :: 45·4 in. : 131·75 + ft. = HA
28·75 = HB
160·50 + = AB.

Exercise 11.

As 84 : 100 :: 65 feet : 77·4 − feet.

Exercise K.

As 100 : 72 :: 93 feet : 66·96 feet.
66·96 + 13 = 79·96.

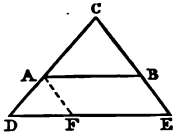
Exercise 12.

As CD : DA :: CE : EB, or
As 29 yards : 12 feet :: 5 fur. − 29 yards, or
As 87 feet : 12 feet :: 3213 feet : 443 + feet.

Exercise L.

As CD = 56 : DA = 15 :: CE = 840 : BE = 225.

Exercise 13.



Drawing AF parallel to BE, we form the Δ DAF similar to ACB. ∴

As DF = 67 : DA = 50 :: AB = 100 : AC = $74\frac{13}{67}$.

Exercise M.

As DF = 85 : DA = 91 :: AB = 100 : AC = 260.

CHAPTER IV.

MENSURATION OF SURFACES.

Problem I.

ANSWERS TO UNRESOLVED EXERCISES.

A. 645·16 sq. feet.
B. $88\frac{3}{4}$ sq. yards.
C. 12 sq. inches.
D. 286 sq. feet, and 1664 pages.
F. $213\frac{3}{8}$ sq. feet.
G. $\frac{81}{242}$ of an acre.
H. £19 : 3 : 3.
K. 87 feet.
L. $2\frac{2}{3}$ inches.
M. $1\frac{2}{3}$ sq. foot.
N. 2·898 + inches.
O. 104·3 + feet.
P. 4·90 − inches.

Exercise 3.

$7\frac{1}{2} \times 5\frac{1}{3} \div \frac{5}{8} \times \frac{16}{3} = \frac{15}{2} \times \frac{8}{1} = 88.$

Exercise 5.

$$26\tfrac{1}{2} \times 12 \times 11\tfrac{3}{4} \div \tfrac{53}{2} \times \tfrac{12}{1} \times \tfrac{47}{4} = \tfrac{53}{2} \times \tfrac{3}{1} \times \tfrac{47}{1}$$
$$= \tfrac{7473}{2} = 3736\tfrac{1}{2} \text{ sq. inches.}$$

This result must be divided successively by 12 and by 12 to reduce it into parts and then into feet.

The same result might have been obtained even more easily by duodecimals, thus:

Ft.	in.
26 :	6
0 :	$11\tfrac{3}{4}$
24 :	3 : 6
1 :	1 : 3
	6 : $7\tfrac{1}{2}$
25 :	11 : $4\tfrac{1}{2}$.

Exercise 6.

$$944 \times 587 \div 554128 \text{ sq. links} = 5{\cdot}54128 \text{ acres.}$$

Exercise 7.

$$9)555025 \text{ sq. ft.} = 745^2.$$
$$61670 - \text{sq. yards.}$$
$$61670 \div 4840 \div 12{\cdot}742 - \text{acres.}$$
$$555025 \div 330 \div 1682 - \text{cottages.}$$

Exercise 8.

$$84 \times 47\tfrac{3}{4} \div 4011 \text{ sq. feet} = 445\tfrac{2}{3} \text{ sq. yds.}$$
$$£{\cdot}066 \times 445\tfrac{2}{3} \div £29{\cdot}414.$$

Exercise 9.

$$(7\tfrac{5}{8})^2 = (\tfrac{61}{8})^2 = \tfrac{3721}{64} \div \text{area of one side.}$$
$$\tfrac{3721}{64} \times 6 = 348\tfrac{17}{32} \div \text{whole area.}$$

Exercise 10.

	Ft.	in.	ft.		ft.		pts.
Top	2	: 6	× 3	=	7	:	6
Side	1	: 4	× 3	=	4	:	0
End	1	: 4	× 2½	=	3	:	4
					14	:	10
							2
				Ans.	29	:	8.

Exercise D.

$$78 \times 3\tfrac{8}{3} \doteq 286.$$
$$9 \times 5\tfrac{1}{2} = 49\tfrac{1}{2}.$$
$$2 \times 286 \times 144 \div 49\tfrac{1}{2} \doteq 1664.$$

Exercise F.

$$12 \times 5\tfrac{1}{2} \doteq 66$$
$$12 \times 2\tfrac{1}{3} \doteq 28$$
$$5\tfrac{1}{2} \times 2\tfrac{1}{3} \doteq 12\tfrac{4}{6}$$
$$106\tfrac{4}{6} \times 2 \doteq 213\tfrac{2}{3}.$$

Exercise G.

$$\frac{45 \times 36}{4840} \doteq \frac{81}{242}.$$

Exercise H.

We have 287 at 9d., making £10 : 15 : 3, and 168 at 1s., making £8, 8s.

Exercise 11.

$$2813 \div 58 \doteq 48\tfrac{1}{58}.$$

Exercise 12.

$1 \times 9 \times 144 \div 9\frac{1}{2} \doteq 136\frac{8}{19}$ inches.

Exercise L.

$15\frac{1}{8} \div 5\frac{3}{4} \doteq \frac{46}{3} \times \frac{4}{23} = \frac{8}{3} = 2\frac{2}{3}$.

Exercise M.

Breadth $\doteq \sqrt{(26^2 - 24^2)} = 10$

Exercise 17.

$\sqrt{(4830{\cdot}25 \text{ sq. in.})} \doteq 69{\cdot}5$ in.

Exercise 18.

70 ft. 10 pts. 1 in. $\doteq$ 10201 sq. inches.
$\sqrt{10201}$ sq. inches $\doteq$ 101 inches.

Exercise 20.

$\sqrt{4840}$ sq. yards $\doteq 69{\cdot}6 -$ yards.

Exercise O.

Quarter of an acre contains 1210 square yards, the square root of which is 34·785 + yards.

Exercise P.

Each side will be one sixth of a square foot = 24 square *inches, the square* root of which is a side of one square, or *an edge of the* cube.

PROBLEM II.

ANSWERS TO UNRESOLVED EXERCISES.

A. 17·71—. | B. 9 ft. 4 pts. 8 in.

EXERCISE 3.

By the reverse formula,

$$\text{Diag.} \doteq \sqrt{(2 \times 578)} = \sqrt{1156} = 34.$$

PROBLEM III.

ANSWERS TO UNRESOLVED EXERCISES.

A. 36. | B. $17\frac{9}{198}$. | C. 9984.

EXERCISE 4.

$$3{\cdot}48 + 8{\cdot}31 \doteq 11{\cdot}79.$$
$$11{\cdot}79 \times 1{\cdot}38 \doteq 16{\cdot}2702 \text{ sq. ch.} = 1{\cdot}62702 \text{ acre.}$$

EXERCISE C.

By drawing a perp. from the obtuse angle upon the opposite side, we cut off a right-angled triangle, of which we have given the hyp. = 100 and the base = 28. From these we find the perp. 96, which is the breadth of the trapezoid.

PROBLEM IV.

ANSWERS TO UNRESOLVED EXERCISES.

A. 14·5145.
B. $33\frac{21}{40}$.
C. 72 ft. 11 p. 4 in.

D. 2574.
E. ·80028 acre.
F. 3873 + sq. yds.

G. 43·45 +.

Exercise 2.

$$29\tfrac{1}{2} \times 33\tfrac{3}{3} \div 2 \doteq {}^{59}/_{2} \times {}^{100}/_{3} \times \tfrac{1}{2} = {}^{5950}/_{12} = 496\tfrac{7}{12}.$$

Exercise 3.

5 ft. 3 in. × 10 in. ≐ 4 ft. 4 pts. 6 in.

Exercise B.

$$18\tfrac{5}{8} \times 3\tfrac{3}{5} \div 2 \doteq {}^{149}/_{8} \times {}^{18}/_{5} \times \tfrac{1}{2} = {}^{1341}/_{40}.$$

Exercise C.

8 ft. 8 in. × 8 ft. 5 in. ≐ 72 sq. ft. 11 pts. 4 in.

Exercise 4.

By Ch. III., Pr. I., Perp. $\doteq \sqrt{(205^2 - 200^2)} = 45$.

Exercise 5.

Here AB ≐ BC, and AD = DC. (Diagram in *Course.*)
Now, in △ ABD, we have AB = 572 in., and AD = 220 in. Hence (by Pr. I. of Ch. III.) BD ≐ 528 in.

$$528 \times 220 \doteq 116160 \text{ sq. in.} = 806\tfrac{2}{3} \text{ sq. ft.}$$

Exercise 6.

In this instance, again, AB ≐ BC, and AD = DC; and the three sides, AB, BC, and AC, are each 3·4. ∴ in △ ADB, we have AB = 3·4, and AD = ½ AC = 1·7. Hence $BD = \sqrt{(AB^2 - AD^2)} \doteq \sqrt{(3{\cdot}4^2 - 1{\cdot}7^2)} = \sqrt{8{\cdot}67} = 2{\cdot}9445 -$

Exercise 7.

By drawing lines from the centre of the hexagon to each of the angular points, we divide it into six equal equilateral triangles, having each side = 1.

In any one of these we find the altitude as in Ex. 6, viz.—

$$P = \sqrt{(1^2 - \cdot 5^2)} = \sqrt{\cdot 75} = \cdot 8660254 + .$$

$\therefore$ Area of one $\Delta \doteq \cdot 4330127 +$,

and area of hexagon $\doteq \cdot 4330127 \times 6$.

Exercise D.

We first find the perpendicular, 117.

Exercise F.

Taking the answer to Exercise E, viz. ·80028 acre, we reduce it to sq. yards by multiplying by 4840.

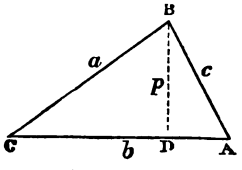
Exercise G.

The octagon being divided into eight triangles by lines drawn from the centre to the several angular points, we have the area of one triangle = half the product of its base, 8, by its altitude, which is the radius of the inscribed circle, viz. 3·621. Therefore the area of the octagon will be four times that product, or 12 times 3·621.

Problem V.

DEMONSTRATION OF THE RULE.

Let ABC be any triangle, of which the longest side, AC, is taken as the base, a perp., BD, being drawn to it from the vertex. Let a, b, and c represent, respectively, the given sides opposite the angles A, B, and C; and let CD and DA, which are not given, be expressed by the *letters* x *and* y. We have (by Euc. I. 47),



$$\text{in } \triangle \text{ADB, } BD^2 = c^2 - y^2,$$
$$\text{and in } \triangle \text{CDB, } BD^2 \doteq a^2 - x^2.$$
$$\therefore c^2 - y^2 \doteq a^2 - x^2;$$
$$\text{or, transposing, } x^2 - y^2 \doteq a^2 - c^2.$$
$$\text{But } x + y \doteq b.$$

Dividing one of these two equations by the other,

$$x - y \doteq \frac{a^2 - c^2}{b}.$$

$$\text{But } x + y \doteq b = \frac{b^2}{b}.$$

Adding these two equations together,

$$2x \doteq \frac{a^2 + b^2 - c^2}{b},$$

$$\text{and } x \doteq \frac{a^2 + b^2 - c^2}{2b} = \text{CD}.$$

$$\therefore \text{CD}^2 \doteq \frac{a^4 + 2a^2b^2 + b^4 - 2a^2c^2 - 2b^2c^2 + c^4}{4b^2}.$$

$$\text{CB}^2 \doteq a^2 = \frac{4a^2b^2}{4b^2}.$$

$$\therefore \text{BD}^2 = \text{CB}^2 - \text{CD}^2 \doteq \frac{2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4}{4b^2};$$

$$\text{and BD} \doteq \frac{1}{2b} \times \sqrt{(2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4)}.$$

$$\text{But, area of } \triangle \text{ABC} \doteq \tfrac{1}{2}(\text{AC} \times \text{BD})$$
$$= \tfrac{1}{4}\sqrt{(2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4)},$$
$$= \sqrt{\left(\frac{a+b+c}{2} \times \frac{a+b-c}{2} \times \frac{a-b+c}{2} \times \frac{-a+b+c}{2}\right)},$$

an expression which is identical with our rule, since the first factor is half the sum of the three sides, and each of the others is that half sum diminished by one of the sides.

NOTE. The reason of selecting the *longest* side as the base, is to avoid a slight variation which would be necessary in the demonstration, if either of the angles A or C were obtuse.

ANSWERS TO UNRESOLVED EXERCISES.

A. 504.
B. 4·901+ acres.
C. 14 ft. 8 pts. 10−in.
D. 8·714+.

Exercise 1.

$$21 \times 6 \times 7 \times 8 \doteq 7056.$$

Exercise 2.

$$3162 \times 737 \times 744 \times 1681 \doteq 2{,}914{,}539{,}881{,}616.$$
$$\sqrt{2{,}914{,}539{,}881{,}616} \doteq 1{,}707{,}202 + \text{sq. links.}$$

Exercise A.

$$54 \times 28 \times 14 \times 12 \doteq 254016.$$

Exercise B.

$$16{\cdot}31 \times 3{\cdot}76 \times 5{\cdot}82 \times 6{\cdot}73 \doteq 2402{\cdot}04 -.$$

Exercise 3.

$$68{\cdot}5 \times 33{\cdot}5 \times 17{\cdot}5 \times 17{\cdot}5 = 702767.$$
$$\sqrt{702767} \doteq 838{\cdot}3 +.$$

Exercise 4.

$$\sqrt{(35 \times 15 \times 14 \times 6)} \doteq \sqrt{44100} = 210.$$
Or, $\sqrt{(35 \times 15 \times 14 \times 6)} \doteq \sqrt{(7 \times 5 \times 5 \times 3 \times 7 \times 2 \times 3 \times 2)}$
$$= \sqrt{(7^2 \times 5^2 \times 3^2 \times 2^2)} = 7 \times 5 \times 3 \times 2 = 210.$$

Exercise 5.

$$1194 \times 398 \times 398 \times 398 \doteq 75{,}275{,}481{,}648.$$
$$\sqrt{75{,}275{,}481{,}648} \doteq 274364 -.$$
$$274364 \div (9 \times 4840) \doteq 6{\cdot}2985 +.$$

Or thus:—

$$\sqrt{(1194 \times 398 \times 398 \times 398)} \doteq \sqrt{(3 \times 398^2 \times 398^2)}$$
$$= \sqrt{3} \times 398^2 = 1{\cdot}73205 \times 158404 = 274364 -.$$

Exercise C.

$$105 \times 35 \times 35 \times 35 \doteq 4501875.$$
$$\sqrt{4501875} \doteq 2122 -.$$

Or thus:—

$$\sqrt{(3 \times 35 \times 35 \times 35 \times 35)} \doteq 35^2 \times \sqrt{3} = 1225 \times 1{\cdot}732 +.$$

Exercise D.

$$6{\cdot}75 \times 2{\cdot}5 \times 2{\cdot}25 \times 2 \doteq 75{\cdot}9375.$$

Problem VI.

ANSWERS TO UNRESOLVED EXERCISES.

A. 1 ac. 2 r. 6 + p. | **B.** 7447 + sq. yds. | **C.** 2·790 − ac.

Exercise 1.

$$\triangle ABC \doteq 278 \times 264 = 78392$$
$$\triangle ADC \doteq 278 \times 235 = 65330$$
$$\text{Quadrilateral } ABCD \doteq \underline{\underline{138722}} \text{ sq. l.}$$

Or thus:—

$$\text{Quad. } ABCD \doteq 278 \times (264 + 235) = 278 \times 499.$$

Exercise 2.

Converting at once the links into chains,

$$9{\cdot}635 \times 3{\cdot}355 \times 4{\cdot}995 \times 1{\cdot}285 \doteq 207{\cdot}483 +.$$
$$10{\cdot}125 \times 2{\cdot}795 \times 5{\cdot}555 \times 1{\cdot}775 \doteq 279{\cdot}035 +.$$
$$\triangle ABC \doteq \sqrt{207{\cdot}483} = 14{\cdot}404 + \text{ sq. chains.}$$
$$\triangle ADC \doteq \sqrt{279{\cdot}035} = 16{\cdot}704 + \ldots\ldots\ldots\ldots$$
$$\text{Trapezium } ABCD \doteq \underline{\underline{31{\cdot}108 + \ldots\ldots\ldots\ldots}}$$

Exercise A.

$3{\cdot}69 \times 2{\cdot}38 \doteq 8{\cdot}7822$
$3{\cdot}69 \times 1{\cdot}79 \doteq 6{\cdot}6051$

$15{\cdot}3873$ sq. chains.

Or, $3{\cdot}69 \times 4{\cdot}17 \doteq 15{\cdot}3873$ sq. chains.

Exercise B.

$15{\cdot}3873 \times 484 \doteq 7447 +$ sq. yards.

Exercise C.

$9{\cdot}58 \times 2{\cdot}13 \times 2{\cdot}85 \times 4{\cdot}60 \doteq 267{\cdot}515 -.$
$7{\cdot}83 \times 2{\cdot}01 \times 2{\cdot}97 \times 2{\cdot}85 \doteq 133{\cdot}217 -.$

Δ ABD $\doteq \sqrt{267{\cdot}515} = 16{\cdot}356 -$ sq. chains.
Δ BCD $\doteq \sqrt{133{\cdot}217} = 11{\cdot}542 -$ .....

Trapezium ABCD $\doteq 27{\cdot}898 -$ .....

Problem VII.

Computation of the column of areas in Table I.

The numbers given in that column are the areas of polygons of not more than twelve sides, the length of the side being 1. They are computed in various ways, thus:—

In the case of the square:—

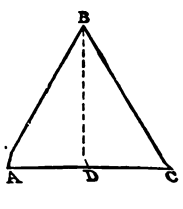
Area of square whose side is $1 \doteq 1 \times 1$ (by Pr. I.) $= 1$.

In the case of the equilateral triangle:—

AB and AC being each 1, AD $\doteq \frac{1}{2}$, and

$BD = \sqrt{(AB^2 - AD^2)}$
$\doteq \sqrt{(1 - {\cdot}5^2)} = \sqrt{({\cdot}75)}$
$= 0{\cdot}866025 +.$

Area of Δ ABC $\doteq \frac{1}{2}$AC $\times$ BD
$= \frac{1}{2} \times 0{\cdot}866025 = 0{\cdot}43301 +.$

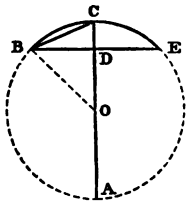


In the case of the hexagon :—

By drawing lines from the centre to each of the angular points the hexagon will be divided into six equilateral triangles, each having its side 1. The area of each triangle, as we have just computed, will be 0·43301 +. Therefore the area of the hexagon is $6 \times \cdot 43301 + = 2 \cdot 5981 -$.

In the case of the dodecagon :—

In the annexed figure let BC be the side of the dodecagon, BE being the side of a hexagon, both inscribed in the circle BCEA. Then if R be put for the radius of the circle, since the side of a hexagon is equal to the radius of the circumscribed circle,



$$BD \doteq \tfrac{1}{2}R, \text{ and}$$
$$OD^2 \doteq OB^2 - BD^2 = R^2 - (\tfrac{1}{2}R)^2 = \tfrac{3}{4}R^2.$$
$$\therefore OD \doteq R \times \tfrac{1}{2}\sqrt{3}, \text{ and}$$
$$CD = OC - OD \doteq R - R \times \tfrac{1}{2}\sqrt{3} = R \times (1 - \tfrac{1}{2}\sqrt{3}).$$

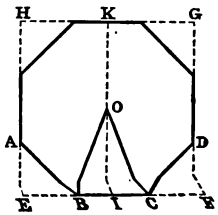
Then, as was demonstrated in Ch. II. Pr. LXXI. Cor. 2,

$$BC^2 \doteq AC \times CD = 2R \times R(1 - \tfrac{1}{2}\sqrt{3}) = R^2 \times (2 - \sqrt{3}).$$
$$\therefore R^2 \doteq \frac{BC^2}{2-\sqrt{3}} = \frac{1}{2-\sqrt{3}} = \frac{1}{2-\sqrt{3}} \times \frac{2+\sqrt{3}}{2+\sqrt{3}} = \frac{2+\sqrt{3}}{4-3}$$
$$= \frac{2+\sqrt{3}}{1} = 2 + \sqrt{3}.$$
$$\text{Dodec.} \doteq 12 \triangle BOC = 6(OC \times BD) = 3OC^2$$
$$= 3R^2 = 6 + 3\sqrt{3} = 6 + 5 \cdot 1962 - = 11 \cdot 1962 -.$$

Cor. Since dodec. $\doteq 3R^2$, the area of any regular dodecagon is equal to three times the square of the radius of the circumscribed circle.

In the case of the octagon :—

Having produced the alternate sides so as to form a square; having found the centre O, and having drawn the other lines in the diagram,



$$\text{Octagon} \doteq 8 \triangle BOC$$
$$= 4(BC \times OI)$$
$$= 2BC \times IK.$$

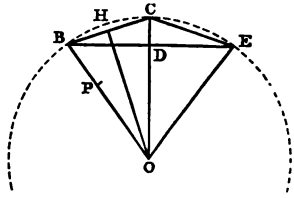
We have now to find IK = EH = EF.
But EF $\doteq$ BC + EB + CF = BC + 2EB.

Now, in $\triangle$ AEB, which is evidently isosceles, we have $AB^2 = AE^2 + EB^2 = 2EB^2$.

$$\therefore 4EB^2 \doteq 2AB^2, \text{ and } 2EB \doteq AB \times \sqrt{2} = \sqrt{2}.$$
$$\therefore EF \doteq 1 + \sqrt{2} = IK, \text{ and}$$
$$\text{Octagon} = 2BC \times IK \doteq 2 \times (1 + \sqrt{2}) = 4{\cdot}8284 + .$$

In the case of the pentagon :—

If OB, the radius of any circle, be divided in the point P, so that $OB \times BP \doteq OP^2$; if the chords BC and CE be placed in the circle, each = OP, and if OC and BE be joined; BC and BE will be the sides of a decagon and pentagon inscribed in the circle.



For we may prove, as in Euc. IV. 10, that, in the Δ BOC, the two angles at B and C are each double of the angle at O. Consequently these three angles are together equal to five times the angle at O. But these three angles are together equal to two right angles. Therefore the $\angle$ BOC $\doteq$ one-fifth of two right angles = one-tenth of four right angles, or one-tenth of all the angular space about the point O. But equal angles at the centre are subtended by equal arcs. Therefore the arc BC is one-tenth of the circumference, and the arc BE(= 2BC) is one-fifth of the circumference. Hence BC and BE are the sides of a decagon and pentagon inscribed in the circle.

Now, to find the numerical relation which BC or OP bears to BO, represent BO by the letter R, and BC or OP by x, and consequently BP by $R - x$.

$$\text{Then, } R \times (R - x) = R^2 - Rx \doteq x^2.$$

Resolving this quadratic equation, we find

$$\text{BC, or } x, = R \times \tfrac{1}{2}(\sqrt{5} - 1).$$
$$\therefore BC^2 \doteq R^2 \times \tfrac{1}{2}(3 - \sqrt{5}).$$

Through O draw OH perp. to BC and consequently

bisecting the line BC and the angle BOC. Then $BH \doteq \frac{1}{2}BC$, and

$$BH^2 = \tfrac{1}{4}BC^2 \doteq R^2 \times \tfrac{1}{8}(3-\sqrt{5}).$$
$$OH^2 = BO^2 - BH^2 \doteq R^2 - R^2.\tfrac{1}{8}(3-\sqrt{5})$$
$$= R^2 \times \left(1 - \frac{3-\sqrt{5}}{8}\right) = R^2 \times \frac{5+\sqrt{5}}{8}.$$
$$\text{But } \angle CBE \doteq \tfrac{1}{2} \angle COE \text{ (Euc. III. 20)}$$
$$= \tfrac{1}{2} \angle BOC = \angle BOH.$$

Therefore the two triangles BOH and CBD are equiangular, and consequently (Euc. VI. 4) have the sides about the equal angles proportional.

$$\therefore BO : OH :: CB : BD,$$
$$\text{and (Euc. VI. 22) } BO^2 : OH^2 :: CB^2 : BD^2,$$
$$\text{or, } R^2 : R^2 \times \frac{5+\sqrt{5}}{8} :: R^2 \times \frac{3-\sqrt{5}}{2} : BD^2.$$
$$\therefore BD^2 \doteq R^2 \times \frac{3-\sqrt{5}}{2} \times \frac{5+\sqrt{5}}{8} = R^2 \times \frac{5-\sqrt{5}}{8}.$$

But, since $BE \doteq 1$, $BD \doteq \frac{1}{2}$, and $BD^2 = \frac{1}{4}$.

$$\therefore \tfrac{1}{4} \doteq R^2 \times \frac{5-\sqrt{5}}{8}, \text{ and } 1 \doteq R^2 \times \frac{5-\sqrt{5}}{2},$$
$$\therefore BO^2 = R^2$$
$$\doteq \frac{2}{5-\sqrt{5}} = \frac{2}{5-\sqrt{5}} \times \frac{5+\sqrt{5}}{5+\sqrt{5}} = \frac{2(5+\sqrt{5})}{25-5} = \frac{5+\sqrt{5}}{10}.$$
$$\therefore OD^2 = BO^2 - BD^2$$
$$\doteq \frac{5+\sqrt{5}}{10} - \tfrac{1}{4} = \frac{5+2\sqrt{5}}{20} = \tfrac{1}{4}(1 + \tfrac{2}{5}\sqrt{5}).$$
$$\therefore BD^2 \times OD^2 \doteq \tfrac{1}{4} \times \tfrac{1}{4}(1 + \tfrac{2}{5}\sqrt{5}) = \tfrac{1}{16}(1 + \tfrac{2}{5}\sqrt{5}).$$
$$\therefore \triangle BOE = BD \times OD \doteq \tfrac{1}{4}\sqrt{(1 + \tfrac{2}{5}\sqrt{5})}, \text{ and}$$
$$\text{Pentagon} = 5 \triangle BOE \doteq \tfrac{5}{4}\sqrt{(1 + \tfrac{2}{5}\sqrt{5})} = \tfrac{5}{4}\sqrt{1{\cdot}894427}$$
$$= \tfrac{5}{4} \text{ of } 1{\cdot}37638 = 1{\cdot}7205 - .$$

In the case of the decagon :—

BE, in the preceding diagram, being the side of a pentagon, and BC the side of a decagon, inscribed in the circle, we previously, in the case of the pentagon, took BE as 1 ; but now we suppose $BC = 1$.

It has been proved above that

$$BC^2 = 1^2 = 1 \doteq R^2 \times \tfrac{1}{2}(3-\sqrt{5}).$$
$$\therefore R^2 \doteq \frac{2}{3-\sqrt{5}} = \frac{2}{3-\sqrt{5}} \times \frac{3+\sqrt{5}}{3+\sqrt{5}} = \frac{3+\sqrt{5}}{2}.$$

It has also been shown that $OH^2 \doteq R^2 \times \frac{5+\sqrt{5}}{8}$.

$$\therefore OH^2 \doteq \frac{3+\sqrt{5}}{2} \times \frac{5+\sqrt{5}}{8} = \frac{5+2\sqrt{5}}{4}.$$

$$\therefore BC^2 \times OH^2$$

$$\doteq 1^2 \times \frac{5+2\sqrt{5}}{4}; \text{ and } BC \times OH \doteq \frac{\sqrt{(5+2\sqrt{5})}}{2}.$$

$$\text{Decagon} = 10 \triangle BOC = 10 \times \frac{BC \times OH}{2} = 5 \times BC \times OH$$

$$\doteq \tfrac{5}{2}\sqrt{(5+2\sqrt{5})} = \tfrac{5}{2}\sqrt{9{\cdot}472136}$$
$$= \tfrac{5}{2} \text{ of } 3{\cdot}07768+ = 7{\cdot}6942+.$$

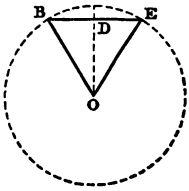
In the case of the heptagon :—

BE being taken as the side of the heptagon, we have the area of the heptagon = 7 times $\triangle BOE = \frac{7}{2}(BE \times OD)$.
But we have, in $\triangle ODB$ (Part III. Ch. VII. Pr. IV.),

$$\text{Radius} : \tan B :: BD = \tfrac{1}{2} : OD.$$

Or, using the natural tangent to radius 1,

$$1 : \tan B :: \tfrac{1}{2} : OD.$$
$$\therefore OD \doteq \tfrac{1}{2} \tan B, \text{ and}$$
$$\text{Heptagon} = \tfrac{7}{2}(BE \times OD)$$
$$\doteq \tfrac{7}{4} \tan B.$$
$$\text{But } \angle BOE \doteq \tfrac{1}{7} \text{ of } 360°;$$
$$\text{and } \therefore \angle BOD \doteq \tfrac{1}{14} \text{ of } 360° = \tfrac{2}{7} \text{ of } 90°.$$
$$\therefore \angle B \doteq \tfrac{5}{7} \text{ of } 90° = 64° \; 17\tfrac{1}{7}'.$$



The tangent of this angle taken from a table of natural tangents, is 2·07652 +.

$$\therefore \text{Hept.} \doteq \tfrac{7}{4} \text{ of } 2{\cdot}07562 = 3{\cdot}6339+.$$

In the case of the undecagon :—

We find, in the same manner as with the heptagon, that

$$OD \doteq \tfrac{1}{2} \tan B, \text{ and } \therefore \text{undecagon} \doteq \tfrac{11}{4} \tan B.$$
$$\text{But } \angle BOD \doteq \tfrac{1}{22} \text{ of } 360° = \tfrac{2}{11} \text{ of } 90°.$$
$$\therefore \angle OBD \doteq \tfrac{2}{11} \text{ of } 90° = 73° \; 38\tfrac{2}{11}'.$$
$$\text{Tan } B \doteq 3{\cdot}40569-.$$
$$\therefore \text{Undecagon} \doteq 9{\cdot}3656+.$$

ANSWERS TO UNRESOLVED EXERCISES.

A. 589·9 −.

B. 14 sq. ft. 105·7 + in.

C. Tetrahedron, 1·732 +.
Hexahedron, 6.
Octahedron, 3·464 +.
Dodecahedron, 20·65 −.
Icosahedron, 8·660 +.

D. 156·1 + square inches.

EXERCISE 2.

$$\text{Area} \doteq \frac{24336 \times 4{\cdot}8284}{9 \times 4840} = \frac{270{\cdot}4 \times 1{\cdot}2071}{121} \text{ ac.}$$

EXERCISE 3.

$$\text{Area} \doteq 389^2 \times {\cdot}43301 = 151321 \times {\cdot}43301$$
$$= 65524 - \text{sq. links} = {\cdot}65524 - \text{ac.}$$

EXERCISE 4.

The area of each end, by the rule, is 5·3128. The sides are rectangles, the area of one of which is 8·5 × 1·43; or the area of all the six is 51 × 1·43 = 72·93.

EXERCISE A.

$$8{\cdot}756^2 \doteq 76{\cdot}667536.$$

EXERCISE 5.

The area of the base, by the rule, is $45^2 \times 3{\cdot}6339 = 7359 -$ sq. inches.

The area of each side is 45 × 32 = 1440 sq. in.

EXERCISE D.

$$\textit{Surface} \doteq (2\tfrac{3}{4})^2 \times 1{\cdot}7205 \times 12 = \tfrac{121}{16} \times 20{\cdot}646.$$

[illegible] VIII.

[illegible]

A. [illegible] — [illegible] B. [illegible] + [illegible]

[illegible] I.

Field ABCDH.		Field DEFGH.	
△ABC =	[illegible] —	△DEF =	[illegible] —
△ACD =	[illegible] —	△DFH =	[illegible] —
△ADH =	[illegible] —	△HFG =	[illegible] —
	[illegible] — sq. ch.		[illegible] — sq. ch.

The *products* before the roots are extracted, for the several triangles, are as follows:—

△ABC. [illegible] — .	△DEF. [illegible] — .
△ACD. [illegible] — .	△DFH. [illegible] — .
△ADH. [illegible] — .	△HFG. [illegible] — .

These products may be useful to check the operation when logarithms are not employed.

EXERCISE A.

Field ABCDH.	Field DEFGH.
△ABC = 0·8292 +	△DEF = 1·1569 −
△ACD = 1·9376 +	△DFH = 2·2178 +
△ADH = 1·9000 −	△HFG = 2·4924 +
4·6668 + ac.	5·8671 + ac.

The *products*, expressing all the figures after the first six by ciphers, and taking the nearest figure for the sixth, are—

△ABC, 6,876,240,000.	△DEF, 13,383,500,000.
△ACD, 37,544,000,000.	△DFH, 49,186,800,000.
△ADH, 36,099,300,000.	△HFG, 62,120,100,000.

EXERCISE 2.

Triangle ABG.....	$\doteq$ 88,812
Triangle GBF.....	= 202 143
Triangle BIC.....	= 10 537
Trapezoid CIKD.....	= 5 824
Trapezoid DKLE.....	= 13 728
Triangle ELF.....	= 14 110
	335,154 sq. links.

The *products*, for the two first triangles, are—

Δ ABG, 7,887,550,000. Δ GBF, 40,861,700,000.

EXERCISE B.

Triangle ABC.....	$\doteq$ 4·5904 – acres.
..... ACD.....	= 7·6142 –
..... *gi*D.....	= 0·4575 –
..... D*kl*.....	= 0·6160 +
Trapezoid *kmnl*.....	= 1·0217 –
Triangle A*mn*.....	= 0·3029 +
	14·6027 –
Deduct Δ C*fg*.....	= 0·3301 –
Total.....	= 14·2726.

Products, for the two large triangles.

Δ ABC, 2107·14 +. Δ ACD, 5797·57 –.

PROBLEM XII.

NOTE. The simple view of the subject, given in the *demonstration* in the text, is undoubtedly sufficient to convince the understanding; but if more rigorous demonstration is called for, we must first show that, "About a circle we may describe a polygon of so many sides that its circumference shall differ from that of the circle by a quantity less than any assignable quantity, and that its area shall also differ from the area of the circle by a quantity less

than any that can be assigned:" but into proof of that it is unnecessary to enter here, since demonstrations may be found in any of our modern treatises on Geometry, to which the subject more properly belongs. (See, for instance, *Playfair's Elements, Sup. B.* I. *Pr.* 4 *and* 5 ;—*Library of Useful Knowledge, Geometry, B.* III. § 4, *Pr.* 31 *and* 32,—*Elements de Geometrie, par Legendre, Liv.* IV. *Pr.* 12.—*Thomson's Euclid, Ap.* I. 39.)

None of these demonstrations, however, can be regarded as *strictly* perfect, since all assume, as an axiom, expressed or implied, that the circumference of a polygon is always greater than that of its inscribed circle.

ALGEBRAICAL DEMONSTRATION.

Let z be an arc of a circle, and s the corresponding sector, both regarded as variable, the radius being constant. Then

$$\partial s \doteq \tfrac{1}{2} r \partial z.$$

$\therefore$, by integrating, $s \doteq \tfrac{1}{2} rz$.

And, when z becomes the whole circumf., s becomes the area of the whole circle, $\therefore$

$$\text{Area of circle} \doteq \tfrac{1}{2}(R \times C).$$

It may be doubted if this or any other demonstration is superior in strictness to the simple one given in the text. They will all, it is believed, if traced to their various sources, be found to involve some consideration of the same subtle and undemonstrated kind as that involved in the first, although, in every case, quite satisfactory for conviction.

ANSWERS TO UNRESOLVED EXERCISES.

A. ·0490875. | B. $170\frac{113}{1280}$.

EXERCISE 2.

$29{\cdot}5 \times 185{\cdot}35 \div 2 \doteq 2734$ — sq. inches.

PROBLEM XIII.

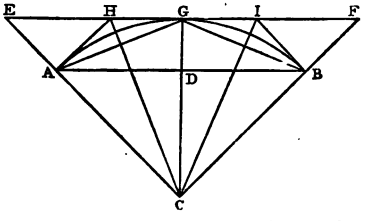
ANOTHER DEMONSTRATION.

The rule may be demonstrated *Geometrically*, independently of the two problems quoted in the text, and the value of π investigated, by what is called the *Method of Exhaustions*, thus:—

In the first place, we know that the area of a circle must be greater than that of any inscribed and less than that of any circumscribed polygon, and that the greater the number of sides of either, the nearer it will approach the area of a circle. Now, in order to find the area of a polygon of a very great number of sides inscribed in a circle, or of one described about it, the operation will be much facilitated by previously resolving the following problem.

PRELIMINARY PROBLEM. *Having given the area of a regular polygon inscribed in a circle, and of a similar polygon described about it, to find the area of an inscribed and of a circumscribed circle of double the number of sides.*

Let AB be the side of the given inscribed polygon; EF, parallel to AB, that of the similar circumscribed polygon; and C, the centre of the circle. If the chord, AG and the tangents, AH, BI, be drawn, the chord, AG will be the side of an inscribed polygon of double the number of sides; and HI (= 2HG), the side of a similar circumscribed polygon. Whatever part then the triangles ACD and ECG are of the two given polygons, the same part are the triangle ACG and the quadrilateral ACGHA, of the two required polygons, since the ratio that each of these bears to the whole polygon is the same as that of the angle ACG to four right angles.



Let M be the area of the given inscribed polygon, and N that of the given circumscribed polygon; and let m and n be the areas of the two required polygons.

Then, *first*, the triangles ACD, ACG, being of the same altitude, are to each other as their bases CD and CG (Euc. VI. 1). And they are also to each other as the polygons M and m, of which they are like parts. Therefore $M : m :: CD : CG$. Again, the triangles CAG, CEG, are to each other as their bases CA and CE, and therefore the polygons m and N (of which these two triangles are like parts) have the same ratio. Therefore $m : N :: CA : CE :: CD : CG$ (Euc. VI. 2, with V. 18) $:: M : m$ (Euc. V. 11). Hence $m^2 \doteq M.N$ (Euc. VI. 16), and $m \doteq \sqrt{(M.N)}$.

Secondly.—The triangles CHG and CHE are to each other as their bases GH and HE. But since, in the $\triangle$ ECG, the vertical angle ECG is bisected by CH, we have $GH : HE :: GC : CE$ (Euc. VI. 3) $:: CD : CA :: CD : CG :: \triangle CAD : \triangle CAG :: M : m$. Therefore $\triangle CHG : \triangle CHE :: M : m$. Hence, by inversion and composition, $\triangle CHG + \triangle CHE = \triangle CGE : \triangle CHG :: M + m : M$; and, doubling the consequents, $\triangle CGE : 2\triangle CHG =$ quad. $CAHG :: M + m : 2M$. Now the $\triangle CGE$ and the quad. CAHG, are like parts of the polygons N and n. Therefore $N : n :: M + m : 2M$; and, consequently, $n \doteq \dfrac{2M \times N}{M + m}$.

Now, to apply these two results to finding the area of the circle.

The area of a square inscribed in a circle having its radius 1, is 2; and that of a circumscribed square is 4. Therefore we have $M = 2$, and $N = 4$, to commence with; and, by the two formulæ found in the preliminary problem above, $m \doteq \sqrt{(M.N)} = \sqrt{(2 \times 4)} = \sqrt{8} = 2{\cdot}8284271$, and

$$n \doteq \frac{2M \times N}{M + m} = \frac{4 \times 4}{2 + 2{\cdot}8284271} = 3{\cdot}3137085.$$

We have thus found the areas of an inscribed and of a circumscribed octagon; and, considering these as now given, we have, for a new operation, $M = 2{\cdot}8284271$, and $N = 3{\cdot}3137085$, and find $m = 3{\cdot}0614674$, and $n = 3{\cdot}1825979$, which are the areas of polygons of sixteen sides. And thus we proceed, finding successively the numbers in the *following table.*

Number of Sides.	Area of Insc. Polygon.	Area of Circ. Polygon.
4	2·000,000,0	4·000,000,0
8	2·828 427 1	3·313 708 5
16	3·061 467 4	3·182 597 9
32	3·121 445 1	3·151 724 9
64	3·136 548 5	3·144 118 4
128	3·140 331 1	3·142 223 6
256	3·141 277 2	3·141 750 4
512	3·141 513 8	3·141 632 1
1024	3·141 572 9	3·141 602 5
2048	3·141 587 7	3·141 595 1
4096	3·141 591 4	3·141 593 3
8192	3·141 592 3	3·141 592 8
16384	3·141 592 5	3·141 592 7
32768	3·141 592 6	3·141 592 6

Hence it appears that the areas of the inscribed and circumscribed polygons of 32768 sides agree to the first eight figures of the numeral expressions for their value: therefore the numeral expression for the circle itself will agree with the polygons to the extent of the same eight figures, so that, the radius being 1, the area is $3{\cdot}141,592,6\pm$, which expresses the number Π as far as the decimal places are carried.

But the areas of circles being to one another as the squares of their radii, it follows that the area of any circle $\doteq R^2 \times \Pi$.

ANSWERS TO UNRESOLVED EXERCISES.

A. 1075 +.	C. 1·735 − ac.	D. 170·87 +.
B. 1810 −.		E. 1·039 − ac.

EXERCISE 2.

$(379^2 = 143641) \times 3{\cdot}1416 \doteq 451263$ − sq. links.

EXERCISE 3.

$R^2 \doteq 12{\cdot}92^2 = 166{\cdot}93 -$.

Exercise 4.

$$(59^2 = 3481) \times \cdot 7854 \doteq 2734 - \text{sq. in.}$$

Exercise 5.

$$\{(\tfrac{101}{11})^2 = 3\tfrac{848}{121}\} \times \cdot 7854 \doteq 236\cdot 8 - .$$

Exercise B.

When the diameter is an even number it is most easy to halve it and proceed by Rule 1.

$$(24^2 = 576) \times 3\cdot 1416 \doteq 1809\cdot 5616.$$

Exercise C.

$$(2\cdot 35^2 = 5\cdot 5225) \times 3\cdot 1416 \doteq 17\cdot 349 + \text{sq. ch.}$$

Exercise D.

$$(14\cdot 75^2 = 217\cdot 5625) \times \cdot 7854 \doteq 170\cdot 874 - .$$

Exercise E.

$$\frac{1600 \times 3\cdot 1416}{4840} \doteq \frac{40 \times 3\cdot 1416}{121} = 1\cdot 0385 + .$$

Problem XIV.

Answers to unresolved exercises.

A. $20\cdot 22 - .$ | B. $0\cdot 7958 - \text{ac.}$

Exercise 1.

By Rule 1, $\frac{1}{4}$ of $(9\frac{1}{5})^2 \doteq \frac{48}{5} \times \frac{48}{5} \times \frac{1}{4} = 21\cdot 16.$

$21\cdot 16 \div 3\cdot 1416 \doteq 6\cdot 735 + .$

By Rule 2,

$$\tfrac{4}{5}^3 \times \tfrac{4}{5}^3 \times \cdot07958 \doteq 84\cdot64 \times \cdot07958 = 6\cdot736-.$$

The former answer is the more correct, 3·1416 being a closer approximation to the true number than ·07958.

Exercise A.

$$(15\cdot94^3 = 254\cdot0836) \times \cdot07958 \doteq 20\cdot220-.$$

Problem XV.

Note 1. A geometrical *demonstration* of Rule 1 has been given in the text, or rather briefly alluded to. The algebraical mode of investigating the same rule, regarding the sector and arc as variable, and the radius as constant, shows that $\partial s \doteq \frac{1}{2}r\partial z$, and consequently $s = \frac{1}{2}rz$.

Note 2. It is almost unnecessary to say that the result can, in no case, be depended upon to a greater number of figures than warranted by the particular rule selected for finding the arc.

Answers to unresolved exercises.

A. 11966·13.	D. 545+.	G. 19·1+.
B. 2·630+.	E. 554(17−).	H. 88·6+.
C. 0·1451−.	F. 553(91−).	

Exercise 2.

Arc, by Pr. IX. of Ch. III.....23·292−.
Radius, by Pr. VI. of Ch. III. 12·414+.

Exercise 3.

Circumf. = 578 × 3·1416, 1815·84.
Arc, by Pr. IX. of Ch. III. 473·17.

Exercise B.

By Rule 1, Circumf. $\doteq 16{\cdot}2609+$.
Arc $\doteq \frac{1}{8}$ of $16{\cdot}2609 = 2{\cdot}0326+$.
Sector $\doteq 1{\cdot}0163 \times 2{\cdot}588 = 2{\cdot}6302-$.

Or thus :—

$$\text{Sector} \doteq \frac{2{\cdot}588 \times 2}{8} \times \frac{2{\cdot}588}{2} \times 3{\cdot}1416 = 2{\cdot}6302-.$$

By Rule 2,

$$\text{Sector} \doteq \frac{2{\cdot}588^2 \times 3{\cdot}1416}{8} = \frac{6{\cdot}697744 \times 3{\cdot}1416}{8}.$$

Exercise C.

By Rule 2, the circumf. being 3·1416 the diam. will be 1, and the area of the circle, $1^2 \times {\cdot}7854 = {\cdot}7854$. Then,

$$\text{As } 360^\circ : 66\tfrac{1}{2}^\circ :: 0{\cdot}7854 : 0{\cdot}14508+.$$

The fourth term may be found either by the usual process or by practice.

Exercise 4.

See the diagram under Ex. 5, on the following page.

Arc, by Pr. XIII. of Ch. III.....$323\frac{1}{3}$.
$CD = \sqrt{(BC^2 - BD^2)}$,.....85·323.

By Pr. VIII. of Ch. III.,

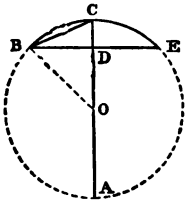
Diam. $\doteq BC^2 \div CD$..... $= 274{\cdot}36$.
$\therefore$ Sector $\doteq 68{\cdot}59 \times 323\frac{1}{3} = 22(177+)$.

Exercise D.

Arc, by Pr. XII. of Ch. III., is 34·07 +.

Exercise 5.

$BC^2 = BD^2 + CD^2$
$= 6^2 + 3^2 = 45.$
$\therefore$ Diam. $= BC^2 \div CD.$
$\doteq 45 \div 3 = 15.$
$BC \doteq \sqrt{45} = 6{\cdot}708+.$
Hence, Arc $\doteq 13{\cdot}89-.$



Exercise 6.

$BD \div BO \doteq 6 \div 7\frac{1}{2} = {\cdot}8000 =$ N. sine of Arc BC.
Hence, Arc to rad. $1 \doteq {\cdot}9273.$
Arc BCE $\doteq {\cdot}9273 \times 7\frac{1}{2} \times 2 = 13{\cdot}9095.$

Exercise E.

$CD \doteq \sqrt{(175^2 - 168^2)} = \sqrt{2401} = 49.$

By Pr. VIII. of Ch. III.,
$AC \doteq BC^2 \div CD = 30625 \div 49 = 625.$
By Pr. XIII. of Ch. III., Arc $\doteq 354{\cdot}\dot{6}.$

Exercise F.

By Pr. XII. of Ch. III., Arc $\doteq 354{\cdot}5.$

Exercise G.

By Pr. XII. of Ch. III., Half arc $\doteq 2{\cdot}76966.$
$2{\cdot}76966 \times 6{\cdot}9 \doteq 19{\cdot}11+.$

Exercise H.

By Pr. VII. of Ch. III., Diam. $\doteq \frac{6{\cdot}72^2}{1{\cdot}96} + 1{\cdot}96 = 25.$
By Pr. XII. of Ch. III., Half arc $\doteq 7{\cdot}09.$
$7{\cdot}09 \times 12\frac{1}{2} \doteq 88{\cdot}625.$

PROBLEM XVI.

DEMONSTRATION OF RULE 6.

If we put c for the chord and h for the height of the segment, d expressing the diameter of the circle, then the area of the segment will be truly expressed by the following series,—

$$4h\sqrt{dh}\times \times \left\{\frac{1}{3}-\frac{1}{5}\cdot\frac{1}{2}\left(\frac{h}{d}\right)-\frac{1}{7}\cdot\frac{1}{2\cdot4}\left(\frac{h}{d}\right)^2-\frac{1}{9}\cdot\frac{1.3}{2.4.6}\left(\frac{h}{d}\right)^3-\&c.\right\}.$$ *

Now since h is always less than d, the fraction $h \div d$ is less than 1, and its ascending powers continually diminish in value as they advance in the series. For a double reason, therefore, the successive terms of the series become less and less, and *that* so rapidly, when h is small compared with d, that the whole value of the series will, in that case, be found to be nearly identical with the sum of two or three leading terms; and in all cases not very far from it. If, therefore, the expression given in Rule 6 can be proved to agree with the above expression as far as two or three terms of the series, it will be shown to give an approximation to the truth (which is all that it professes), and *that* the more close by how much the smaller the segment is.

To show that the two expressions do agree to a certain extent, we have, by Euc. III. 35 (referring to the diagram in the *Course*), $BD^2 = CD \times DA$, or

$$\tfrac{1}{4}c^2 = h(d-h) = hd - h^2.$$
$$\therefore \tfrac{1}{4}c^2 + \tfrac{4}{10}h^2 \doteq hd - h^2 + \tfrac{4}{10}h^2 = hd - \tfrac{3}{5}h^2.$$
$$\therefore \tfrac{4}{3}h \times \sqrt{(\tfrac{1}{4}c^2 + \tfrac{4}{10}h^2)} \doteq \tfrac{4}{3}h \times \sqrt{(hd - \tfrac{3}{5}h^2)}$$
$$= 4h\sqrt{hd} \times \frac{1}{3}\sqrt{\left(1-\frac{3}{5}\cdot\frac{h}{d}\right)}$$
$$= 4h\sqrt{hd} \times \left\{\frac{1}{3}-\frac{1}{5}\cdot\frac{1}{2}\left(\frac{h}{d}\right)-\frac{3}{25}\cdot\frac{1}{2.4}\left(\frac{h}{d}\right)^2-\&c.\right\},$$

as is found by extracting the latter root in the previous member of the equation.

Now the series just found agrees, it will be observed, with

* *The demonstration* of this may be found in the Key to the Author's *Complete Treatise* (Part IV., Pr. XVI., Rule III.), and in various other works.

the true series given above, in two terms, and differs in the third term only by having $\frac{3}{25}$ instead of $\frac{1}{7}$ or $\frac{3}{21}$, giving an error of $+\frac{1}{350}\left(\frac{h}{d}\right)^2$ in that term, a fraction which, as well as all the subsequent terms, is trifling compared with the values of the previous terms when h is small compared with d.

The total error, in employing Rule 6, will not be more than 1 in 23000 if the segment does not exceed 60°; or 1 in 4560, if not exceeding 90°. But it will amount to $1-1360^{\text{th}}$ part of the whole if the segment is of 120°, and will exceed $1-231^{\text{st}}$ part if the segment is greater than a semicircle. This rule gives a result always greater than the truth.

The fraction $\frac{4}{10}$ is not reduced to its lowest terms, because $\frac{4}{10}$ is an easier multiplier than $\frac{3}{8}$. For a similar reason the factor 2, external to the root, is not permitted to pair off with $\frac{1}{4}$ internal to the radical sign, since $\frac{1}{4}c^2$ is often as easily found as c^2.

ANSWERS TO UNRESOLVED EXERCISES.

A. 71·0+.
B. 0·1683−.
C. 44(5·7+).
D. 446·(5−).
E. 919·(9+).
F. 100(9−).
G. 1013+.

EXERCISE 1.

By Rule 1,

Area of Circle.....	$\doteq$ 1809·56+.
..... Sector.....	452·39+.
..... Triangle.....	288·00.
..... Segment.....	164·39+.

By Rule 5. Looking for 90° in the right-hand column of Table II., we find, in the same line, in the column of *segments*, ·2854. This we multiply by 24^2 or 576.

Exercise 2.

By Rule 1.

Area of Circle..... $\doteq \cdot 07958-$.
Diam. of Circle..... $= \cdot 31831-$.
Area of Sector..... $= \cdot 01326+$.
..... Triangle..... $= \cdot 01097-$.
..... Segment..... $= \cdot 00229+$.

By Rule 5. Tabular seg. of $60° \doteq \cdot 0906$. This, multiplied by *radius*^{2} or $\cdot 02533$, gives $\cdot 002295-$.

Exercise B.

$$\text{Segment} \doteq \cdot 00831 \times 20 \cdot 25 = \cdot 1682725.$$

Exercise 3.

Tabular N. sine $\doteq 12 \div 12\frac{1}{2}$..... $= \cdot 9600$.
Nearest on Table,..... $\cdot 9596$.
Corresponding tab. Segment,.... $1 \cdot 0159$.
Required Seg. $\doteq 1 \cdot 0159 \times (12\frac{1}{2})^2 = 158 \cdot 7+$.

Exercise C.

By Rule 5. $\text{Segment} \doteq \cdot 1141 \times 3906 \cdot 25$.

Exercise D.

$$\text{OD} \doteq \sqrt{(62 \cdot 5^2 - 33 \cdot 6^2)} = \sqrt{2777 \cdot 29} = 52 \cdot 7.$$
$$\text{CD, or } h, \doteq 62 \cdot 5 - 52 \cdot 7 = 9 \cdot 8.$$
$$\tfrac{1}{4}c^2 \doteq 1128 \cdot 960$$
$$\tfrac{4}{10}h^2 = \quad 38 \cdot 416$$
$$\sqrt{1167 \cdot 376} = 34 \cdot 167-.$$
$$34 \cdot 167 \times \tfrac{1}{3} \times 9 \cdot 8 \doteq 446 \cdot 488.$$

Exercise 4.

Tabular N. V. S. $\doteq 2{\cdot}7 \div 3{\cdot}75 \ldots\ldots = {\cdot}7200$.
Nearest on Table, ... ·7188.
Corresp. tabular Segment, ... 1·0159.
Req. Segment $\doteq 1{\cdot}0159 \times 14{\cdot}0625 = 14({\cdot}29-)$.

Exercise E.

Proceeding by the second formula in Rule 2,

$$hd \doteq 1344$$
$$\tfrac{6}{10}h^2 \doteq \quad 264{\cdot}6$$
$$\sqrt{1079{\cdot}4} \doteq 32854+.$$
$$32854 \times (\tfrac{2}{3}h = 28) \doteq 919{\cdot}912.$$

Exercise 5.

$AD \doteq 10^2 \div 4 = 25$. (See diagram in *Course*.)
$\therefore$ Diam. $\doteq 25 + 4 = 29$, and rad. $= 14\frac{1}{2}$.
N. Sine $\doteq 10 \div 14\frac{1}{2} \ldots\ldots = {\cdot}6897$
Nearest on Table, ... ·6905
Corresponding tabular Seg ..... ·2627
Required Seg. $\doteq {\cdot}2627 \times 210{\cdot}25 = 55{\cdot}(23+)$.

Exercise 6.

$c \doteq 20$, and $h \doteq 4$.
$$\tfrac{1}{4}c^2 \doteq 100$$
$$\tfrac{4}{10}h^2 \doteq \quad 6{\cdot}4$$
$$\sqrt{106{\cdot}4} \doteq 10{\cdot}3150.$$
$$10{\cdot}315 \times 4 \times \tfrac{4}{3} \doteq 55{\cdot}0135.$$

Exercise F.

By Pr. VII. of Ch. III., $AD \doteq \dfrac{42^2}{17{\cdot}5} = 100{\cdot}8$.
$\therefore$ Diam. $\doteq 118{\cdot}3$, and radius $= 59{\cdot}15$.
Nat. Sine ..... $\doteq 0{\cdot}7101-$.
Nearest tab. Segment..... $\doteq 0{\cdot}2883$.

Since the nat. sine, ·7101, is nearly half way between

the two that come nearest to it on the table, we should have got an answer nearer the truth if we had taken a segment intermediate to ·2883 and ·2913.

Exercise G.

By Rule 6.

$$\tfrac{1}{4}c^2 \doteq 1764$$
$$\tfrac{4}{10}h^2 \doteq 122{\cdot}5$$
$$\sqrt{1886{\cdot}5} \doteq 43{\cdot}434-.$$

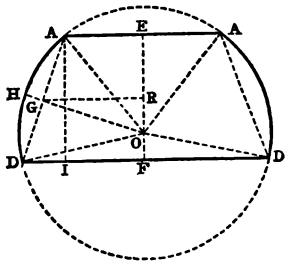
Problem XVII.

ANSWERS TO UNRESOLVED EXERCISES.

A. 7909+.
B. 1579−.
C. 16·9+.
D. 3047+.

Exercise 1.

$AD \doteq \sqrt{(AI^2 + DI^2)} = \sqrt{(21^2 + 3^2)} = \sqrt{450} = 21{\cdot}2132.$
$DG \doteq \tfrac{1}{2}AD = 10{\cdot}6066.$
$DO \doteq \sqrt{(DF^2 + OF^2)}$
$= \sqrt{(144 + 81)} = 15.$



Then finding the segment AHD, by Rule 5 of the last Problem,

N. Sine $\doteq DG \div DO = {\cdot}70705.$
Nearest on Table,... ·7071.
Corresp. tab. Seg.... ·2854.

$$\left.\begin{aligned} {\cdot}2854 \times 225 &\doteq 64{\cdot}215 \\ 64{\cdot}215 & \end{aligned}\right\} = \text{Segments.}$$
$$\tfrac{1}{2}(42 \times 21) = 441{\cdot}000 = \text{Trapezoid.}$$
$$569{\cdot}430 = \text{Zone.}$$

Exercise A.

Radius $\doteq 65$. Chord AD $\doteq 64{\cdot}498$. Each segment (by *Rule 5*) $\doteq 370{\cdot}5+$, and trapezoid $= 7168$.

Exercise B.

Radius $\doteq 65$. Chord AD $\doteq 16{\cdot}125$. Each segment $\doteq 5{\cdot}49$, and trapezoid $= 1568$.

Exercise 2.

GH $\doteq 10 - 7\frac{1}{2} = 2\frac{1}{2}$, since, in this case, OG is parallel to AA or DD, and equal to the half of either.

OF $\doteq \sqrt{(10^2 - 7{\cdot}5^2)}$..... $= 6{\cdot}6144$.
EF $\doteq 2$OF..... $= 13{\cdot}2288$.
Nat. V. Sine $\doteq 2{\cdot}5 \div 10$..... $= {\cdot}2500$.
Hence tabular Segment..... $\doteq {\cdot}2267$.

Segments $= 22{\cdot}67 \times 2$..... $\doteq 45{\cdot}34$
Trapezoid..... $\doteq 198{\cdot}43$

Zone..... $\doteq 243{\cdot}77$ sq. in.

Exercise C.

OE or OF $\doteq 2{\cdot}0488-$. Each segment $\doteq 3{\cdot}34-$, and trapezoid $= 10{\cdot}24+$.

Exercise 3.

Area of Circle $\doteq 150^2 \times \frac{1}{4}\Pi = 17671{\cdot}5$.

But AOA is an equilateral triangle: therefore the $\angle$ AOA is of 60°, and so also is each of the angles AOD and AOD, DOD being, in this instance, a straight line. Therefore the two sectors AOD, AOD, are each one-sixth part of the circle, and the two together are one-third of the circle.

$\therefore$ Sectors AOD, AOD $\doteq 5890{\cdot}5$.
Triangle AOA..... $\doteq 2435{\cdot}7$.

Zone*..... $\doteq 8326{\cdot}+$.

* In this solution a slight deviation has been made from the rule. It was evidently needless to subtract the triangles from the sectors, in order to find the segments, *since the* same triangles were again to be added as parts of the *trapezoid.*

Exercise D.

The most easy method, in this instance, is to find the area of the segment AKA, and subtract it from that of the semicircle.

$$OE \doteq \sqrt{(50^2 - 37{\cdot}5^2)} = \sqrt{1093{\cdot}75} = 33{\cdot}072.$$
$$EK \doteq 50 - 33{\cdot}072 = 16{\cdot}928.$$
$$\text{Nat. v. sine} = 16{\cdot}928 \div 50 \doteq {\cdot}33856.$$

This versed sine is found on the table to be about equidistant from ·3374 and ·3396. We therefore take, for the tabular segment, an arithmetical mean between ·3502 and ·3535, viz. ·35185. Then

$$\text{Semicircle} = 2500 \times 3{\cdot}1416 \div 2 \doteq 3927$$
$$\text{Segment AKA} = {\cdot}35185 \times 2500 \doteq \underline{\ 879{\cdot}625}$$
$$\text{Zone} \ldots\ldots\ldots\ldots \doteq 3047{\cdot}375.$$

Problem XVIII.

Note. No *demonstration* of the rule is necessary. The formulæ are obtained from the rule thus:—

Calling the area of the outer circle A, and that of the inner circle a, the area of any ring will be $A - a$.

Now, by Pr. xiii., $A \doteq R^2 \times \Pi$, and $a \doteq r^2 \times \Pi$.

$$\therefore A - a \doteq (R^2 - r^2) \times \Pi.$$

Again, by Pr. xiv., $A \doteq C^2 \times \frac{1}{4\Pi}$, and $a \doteq c^2 \times \frac{1}{4\Pi}$.

$$\therefore A - a \doteq (C^2 - c^2) \times \frac{1}{4\Pi}.$$

Lastly, since $C^2 - c^2 \doteq (C+c) \times (C-c)$
$$= (C+c) \times (2R\Pi - 2r\Pi),$$

$$A - a \doteq (C+c) \times (R-r)2\Pi \times \frac{1}{4\Pi} = \tfrac{1}{2}(C+c) \times (R-r).$$

ANSWERS TO UNRESOLVED EXERCISES.

A. 180·6 +. | B. £581·4 +.

Exercise 1.

By the first Formula,

$$\text{Area} \doteq (5^2 - 4^2) \times \Pi = 9 \times \Pi = 28{\cdot}274 +.$$

Exercise A.

$$R \doteq 12{\cdot}75, \text{ and } R^2 = 162{\cdot}5625$$
$$r \doteq 10{\cdot}25, \text{ and } r^2 = 105{\cdot}0625$$
$$R^2 - r^2 \doteq 57{\cdot}5.$$

Exercise 2.

By the second Formula,

$$\text{Area} \doteq (400^2 - 357^2) \times {\cdot}07958$$
$$= 32551 \times {\cdot}07958 = 2590 +.$$

Exercise B.

$$R \doteq 135,\ r = 70,\ R^2 - r^2 = 13325,$$
$$13325 \times \Pi = 41862 - \text{sq. ft.} = 4651\tfrac{1}{3} \text{ sq. yds.}$$
$$£0{\cdot}125 \times 4651{\cdot}\dot{3} \doteq £581{\cdot}41\dot{6}.$$

Problem XIX.

Exercise 1.

The easiest mode of computing the segments is by Rule 6 of Problem XVI.

Outer Segment.	Inner Segment.
$\frac{1}{4}c^2 \doteq 24^2 = 576$	$\frac{1}{4}c^2 \doteq 576$
$\frac{4}{10}h^2 \ldots\ldots \doteq 40$	$\frac{4}{10}h^2 = 19{\cdot}6$
$24{\cdot}8193 \doteq \sqrt{616}.$	$24{\cdot}4049 \doteq \sqrt{595{\cdot}6}.$
$24{\cdot}8193 \times \frac{4}{3}$ of $10 \doteq 330{\cdot}9$	$24{\cdot}4049 \times \frac{4}{3}$ of $7 \doteq 227{\cdot}8.$
$227{\cdot}8$	
Ans. $103{\cdot}1.$	

Exercise A.

Outer Segment.	Inner Segment.
$\frac{1}{4}c^2 = 7{\cdot}2^2 \doteq$ 51·84	$\frac{1}{4}c^2 \doteq$ 51·84
$\frac{4}{10}h^2$..... $\doteq$ 3·60	$\frac{4}{10}h^2 \doteq$ 1·764
7·4458.... $\doteq \sqrt{}$ 55·44.	7·3215 $\doteq \sqrt{}$ 53·604.
7·4458 × $\frac{4}{3}$ of 3 $\doteq$ 29·7832	7·3215 × $\frac{4}{3}$ of 21 $\doteq$ 20·5002.
20·5002	
Ans. 9·2830.	

Problem XX.

DEMONSTRATION OF THE RULE.

Since the sides of a right *rectilineal prism* are all rectangles having the same length L, but different breadths; call their breadths D, E, F, &c. Then their areas will be $D \times L$, $E \times L$, $F \times L$, &c.; and their united area will be $(D + E + F + \text{&c.}) \times L = C \times L$. In the case of the right *cylinder or curvilineal prism*, the number of rectangles is infinite, but the demonstration the same.

ANSWERS TO UNRESOLVED EXERCISES.

A. 6·061+. | B. 78·54−.

Exercise 2.

Circumf. $\doteq$ 32 in. × π..... = 100·531

Convex surface ..... = 9047·79
Area of one end $\doteq 16^2 \times \pi =$ 804·25
Area of the other end..... $\doteq$ 804·25

10656·29 sq. in.

Exercise 3.

Each end $\doteq 2{\cdot}0449 \times 2{\cdot}5981 = 5{\cdot}3128$.
Six sides $\doteq 6 \times 8{\cdot}5 \times 1{\cdot}43 = 51 \times 1{\cdot}43 = 72{\cdot}93$.

Exercise A.

Area of Sides,.....5·8860.
Area of ends,... .....0·1754.

Exercise B.

The convex surface is 39·270, and the united area of the ends, the same.

Problem XXII.

Note. The *demonstration*, in the case of the pyramid, may be expressed in more precise algebraical language, in the same manner as the dem. of Pr. xx., above. The same proof may also be extended to the cone, by regarding it as a pyramid of an infinite number of sides.

ANSWERS TO UNRESOLVED EXERCISES.

A. 140·91. | B. 282·7+. | C. 260.

Exercise 2.

$54\frac{5}{12} \times 50 \doteq \frac{653}{12} \times \frac{50}{1} = 2721 -$ sq. feet.

Exercise 3.

Half circumf. $\doteq 2{\cdot}9 \times \Pi = 9{\cdot}1106+.$
$9{\cdot}1106 \times 7{\cdot}35 \doteq 66{\cdot}96+.$

Exercise 4.

Slant height $\doteq \sqrt{(20^2+15^2)} = \sqrt{625} = 25.$
Conv. surf. $\doteq 15\Pi \times 25 = 375\Pi = 1178{\cdot}1 -$ sq. in.
Base $\doteq 30^2 \times {\cdot}7854..... = 706{\cdot}9 - .....$
Whole surface.....$\doteq 1885{\cdot}0 - .....$

Exercise B.

Slant height $\doteq 13$. Conv. surf. $\doteq 204{\cdot}20+$. Area of base $\doteq 78{\cdot}54-$.

Problem XXIII.

Answers to unresolved exercises.

A. 432·9. | B. 21·36−. | C. 942·5 − sq. in.

Exercise 1.

$$3\tfrac{5}{8} + 2\tfrac{3}{4} \doteq 6\tfrac{3}{8}.$$
$$6\tfrac{3}{8} \times 5\tfrac{3}{8} \div 2 \doteq \tfrac{51}{8} \times \tfrac{43}{8} \times \tfrac{1}{2} = 17{\cdot}13+.$$

Exercise 2.

Circumf. of base $\doteq 26{\cdot}075$
... . of top $\doteq 18{\cdot}850$

$\frac{1}{2}$ of $7{\cdot}4 \times 44{\cdot}925 \doteq 166{\cdot}22+$.

Or, more easily thus:—

By the first formula for the frustum of a cone,
$$\text{Area} \doteq \tfrac{1}{2}(8{\cdot}3 + 6) \times 7{\cdot}4 \times \pi = 14{\cdot}3 \times 3{\cdot}7 \times \pi.$$
$$= 52{\cdot}91\,\pi = 166{\cdot}22+.$$

Exercise 4.

$$\text{Area} \doteq (15 + 5) \times 6 \times 74 \div 2 = 60 \times 74.$$

Exercise B.

Area of sides,.....929292 sq. feet.
..... of summit,..... 1024.....
Total,.....930316.....

EXERCISE C.

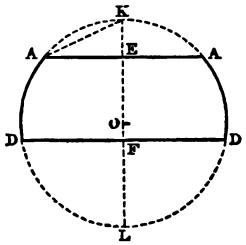
$$\text{Slant height} \doteq \sqrt{(6^2+8^2)} = \sqrt{100} = 10.$$
$$C \doteq 36 \times \Pi. \quad c \doteq 24 \times \Pi.$$
$$\frac{C+c}{2} \doteq 30 \times \Pi = 94{\cdot}248.$$

PROBLEM XXVI.

DEMONSTRATION OF THE RULES.

The rules for this and the subsequent problem may be demonstrated thus:—

Let R be the radius of the sphere; D, its diameter; C, its circumf; s, the surface of a segment AKA, and z the arc AK. Then $AE \doteq \sin z$, and $KE = \text{versin } z$: and, by a principle familiarly employed in the application of the Integral Calculus to the quadrature of surfaces bounded by curves,



$$\partial s \doteq 2\Pi \sin z \partial z$$
$$= 2\Pi R \partial \text{ versin } z,$$
$$\text{Since } R\partial \text{ versin } z \doteq \sin z \partial z.$$
$$\therefore s \doteq 2\Pi R \text{ versin } z,$$

the constant being $= 0$. Therefore, since $2\Pi R \doteq C$,

$$\text{Surface of segment AKA} \doteq C \times \text{KE, and}$$
$$\text{Surface of segment DKD} \doteq C \times \text{KF.}$$
$$\therefore \text{Surf. of Zone AADD} \doteq C \times (\text{KF} - \text{KE}) = C \times \text{EF.}$$

Again, let z become the semicircumf. Then versin z becomes the diameter, and s becomes the surface of the whole sphere.

$$\therefore \text{Surface of sphere} \doteq C \times D = D^2 \times \Pi.$$

COR. 1. Hence the surface of any zone of a sphere is equal to that of a segment of equal height.

COR. 2. The surfaces of any two zones of a sphere, of equal heights, are equal.

COR. 3. The surfaces of segments and zones of the same sphere are to one another as their heights.

Cor. 4. The surface of a sphere is equal to four times the area of a great circle, and the convex surface of a hemisphere is double the area of its base.

Cor. 5. The surface of any segment (AKA) of a sphere is equal to the area of a circle whose radius is the chord (AK) of half the arc of the segment. For $C \times KE \doteq D \times \Pi \times KE = AK^2 \times \Pi$.

Cor. 6. The surface of a sphere is equal to the convex surface of the circumscribed cylinder; and planes, perp. to the axis of the circumscribed cylinder, cut off or intercept equal surfaces on the sphere and cylinder.

ANSWERS TO UNRESOLVED EXERCISES.

A. 63·62 −.	C. 4976 +.
B. 19·635 −.	D. 1·032 −.

Exercise 3.

By the 2d rev. formula of Pr. VI., of Ch. III., Diam. $\doteq 1 \times \cdot 31831$. Therefore the operations by the two rules are in substance the same.

Exercise D.

$2 \cdot 546^2 \doteq 6 \cdot 482116$.

Problem XXVII.

ANSWERS TO UNRESOLVED EXERCISES.

A. 97·56 + sq. ft.	D. 75·40 − sq. in.
B. 188·5 −.	226·20 −.....
C. 706·9 − sq. in.	150·80 −.....

Exercise 1.

Circumf. of sphere $\doteq 82 \cdot 38 \times 3 \cdot 1416 = 258 \cdot 805 +$

EXERCISE 2.

Height of zone $\doteq 59-(12+14)=33$ in.
Circumf. of sphere $\doteq 59\pi = 185{\cdot}354$ in.

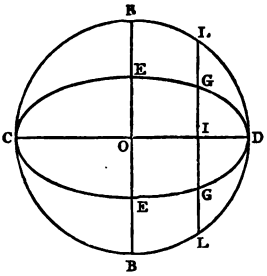
EXERCISE C.

Diam. of sphere $\doteq (12^2 \div 9)+9 = 25$.

PROBLEM XXX.

DEMONSTRATION OF THE RULE.

The rule may be demonstrated *on Geometrical principles*, thus :—



Let a circle be described on one of the axes (CD) of the ellipse, as a diameter; and let a st. line LL move from D to C, keeping parallel to EE, the other axis. In every part of its progress the circular double ordinate, LL, will be to the elliptic double ordinate, GG, as BB to EE. Therefore the circle will have that same ratio to the ellipse, both being generated by the motion of the corresponding double ordinates in the same time; and any segment (LDL) of the circle, cut off by the double ordinate, will have the same ratio to the corresponding segment (GDG) of the ellipse.

If the idea of motion be objected to, suppose the circle to consist of an infinite number of infinitely narrow rectangles parallel to BB, each of which may be represented by the line LL: the ellipse will be made up of an infinite number of rectangles, each of which may be represented by GG. But every one of these rectangles in the circle will be to the corresponding rectangle in the ellipse in the uniform ratio of LL to GG. Therefore the whole circular space, made up of the former class of rectangles, will bear the same ratio to the elliptic space, made up of the same number of the latter class of rectangles, that CD bears to EE.

Now, allowing $2a$ to represent the axis on which the circle is described, viz. CD, and $2b$ the other axis, EE. Then

$$\text{Circle} : \text{Ellipse} :: 2a : 2b.$$
$$\therefore \text{Ellipse} \doteq \frac{b}{a} \times \text{Circle} = \frac{b}{a} \times a^2\Pi = ab\Pi.$$

Or, by the higher Calculus, thus:—

Let s represent the elliptic segment, GDG; s, the corresponding circular segment, LDL; and v the variable abscissa, ID. Then by a principle well known in the application of the Integral Calculus to the quadrature of surfaces bounded by curve lines.

$$\partial s \doteq \text{LL} \times \partial v, \text{ and } \partial s \doteq \text{GG} \times \partial v.$$
$$\therefore \partial s : \partial s :: \text{LL} : \text{GG}.$$

But, by a familiar property of the ellipse,

$$\text{LL} : \text{GG} :: \text{BB} : \text{EE} :: 2a : 2b :: a : b.$$
$$\therefore a : b :: \partial s : \partial s.$$
$$\therefore \partial s \doteq \frac{b}{a} \times \partial s.$$

Integrating both members of the equation, and keeping in view that there is no constant, since both segments vanish together when v becomes 0, we have

$$s = \frac{b}{a} \times s.$$

When v becomes = CD, then s becomes the whole circle, and s the whole ellipse.

$$\therefore \text{Ellipse} \doteq \frac{b}{a} \times \text{circle} = \frac{b}{a} \times a^2\Pi = ab\Pi = 2a \times 2b \times \frac{\Pi}{4}.$$

Exercise 2.

$$\text{Area} \doteq 1{\cdot}25 \times {\cdot}74 \times \Pi = {\cdot}925 \times \Pi = 2{\cdot}906 - \text{sq. ch.}$$

Exercise A.

$$\text{Area} \doteq 345 \times 3\tfrac{1}{2}^5 \times {\cdot}7854 = 59512{\cdot}5 \times {\cdot}7854$$
$$= 46741 + \text{sq. ch.} = 0{\cdot}4674 + \text{acre, Ans.}$$

Problem L.

ANSWERS TO UNRESOLVED EXERCISES.

A. $1\frac{1}{14}$ yards.
B. 676 yards.
C. 1200 sq. in.
D. 116 sq. ft. 86 in.

Exercise 1.

$125^2 : 80^2 :: 2 \text{ ac. } 23\frac{3}{4} \text{ pls.} : 3 \text{ ro. } 20\frac{4}{5} \text{ pls.}$

Exercise 2.

$100^2 : (17\frac{4}{11})^2 :: 31416 : \text{——}.$
$(17\frac{4}{11})^2 \doteq (\frac{191}{11})^2 = 36\frac{481}{121}.$
$36\frac{481}{121} \times 31416 \doteq 36\frac{481}{11} \times 2856 = 947{\cdot}2 -.$

Exercise 3.

$50^2 : 40^2 :: 0{\cdot}25 \text{ ac.} : 0{\cdot}16 \text{ ac.}$

Exercise 4.

$10^2 : 72^2 :: £5{\cdot}5 : £285{\cdot}12.$
$£285{\cdot}12 - £5{\cdot}5 \doteq £279{\cdot}62.$

Exercise B.

$$7^2 : (\tfrac{91}{8})^2 :: 144 : \frac{91 \times 91 \times 144}{7 \times 7 \times 6 \times 6} = 13 \times 13 \times 4.$$

Exercise C.

$(363 \times 12)^2 : 50^2 :: 1{\cdot}452 \times 4840 \times 9 \times 144,$
Or, $121 \times 121 : 50 \times 50 :: 1{\cdot}452 \times 4840,$
Or, $1 : 50 \times 50 :: {\cdot}012 \times 40 : 1200.$

Exercise D.

As $530^2 : 265^2 :: 67160 : \text{area in sq. in.}$
Or, $4 : 1 :: 67160 : 16790$ sq. in.

CHAPTER V.

MENSURATION OF SOLIDS.

PROBLEM I.

ANSWERS TO UNRESOLVED EXERCISES.

A. 18 c. ft. 11 p. 6 s. 8 in.
B. 65 c. ft. 2 pts. 10 sec.
C.
D. 300 tons.
E. 1578−.
F. 15·42−.
G. 8·3.
H. 2·38+.

EXERCISE 1.

$1{\cdot}25^3 \doteq 1{\cdot}953125$ c. foot.
$\cdot 953125 \times 1728 \doteq 1647$ c. in.

Or thus:—

$$(1\tfrac{1}{4})^3 \doteq (\tfrac{5}{4})^3 = \tfrac{125}{64} = 1\tfrac{61}{64} = 1\tfrac{1647}{1728}.$$

EXERCISE 2.

Ft.	in.	
3	: 4	
×3	: 4	
10	: 0	
1	: 1	: 4
11	: 1	: 4
×3	: 4	
33	: 4	: 0
3	: 8	: 5 : 4
37	: 0	: 5 : 4

Or thus:—

$(3\tfrac{1}{3})^3 \doteq (\tfrac{10}{3})^3 = \tfrac{1000}{27}$.

Ft.
3)1000
9)333 : 4
37 : 0 : 5 : 4.

Exercise 3.

$$66 \times 12 \times 12 \doteq 9504 \text{ c. feet.}$$
$$9504 \times 180 \doteq 1710720 \text{ lb.} = 15274 + \text{cwt.}$$

Or, more concisely thus:—

$$\frac{66 \times 12 \times 12 \times 180}{28 \times 4 \times 20} \doteq \frac{66 \times 3 \times 3 \times 9}{7} = 764 - \text{tons.}$$

Exercise D.

$$\text{Weight} \doteq \frac{48 \times 10 \times 8 \times 175}{112 \times 20} = 300 \text{ tons.}$$

In such operations the result is obtained more easily by repeated cancellings.

Exercise 5.

$$\text{Area of base} \doteq 9 \times 2{\cdot}5981 = 23{\cdot}383 -.$$

Exercise 6.

$$\text{Area of base} \doteq (1\tfrac{1}{4})^2 \times {\cdot}7854 = \tfrac{25}{16} \times {\cdot}7854 = 1{\cdot}2272 -.$$

Exercise 7.

$$A \doteq \sqrt{(4{\cdot}5 \times 2{\cdot}5 \times 1{\cdot}5 \times 0{\cdot}5)} = \sqrt{8{\cdot}4375} = 2{\cdot}90574 -.$$

Exercise E.

$$S \doteq 8{\cdot}75^2 \times {\cdot}7854 \times 26{\cdot}24$$
$$= 76{\cdot}5625 \times {\cdot}7854 \times 26{\cdot}24 = 2009 \times {\cdot}7854.$$

Exercise F

$$A \doteq {\cdot}96^2 \times 2{\cdot}5981 - = {\cdot}9216 \times 2{\cdot}5981 - = 2{\cdot}39441 -.$$

Exercise 8.

By the reverse formula for the cube, $L \doteq \sqrt[3]{15{\cdot}625} = 2{\cdot}5$.

Exercise 9.

Content of smaller cube $\doteq 27$.
Content of larger cube $\doteq 54$.
$\sqrt[3]{54} \doteq 3{\cdot}78-$.

Exercise H.

$$\sqrt[3]{2\tfrac{1}{2}} = \sqrt[3]{13{\cdot}500} \doteq 2{\cdot}38+.$$

Problem II.

The *general demonstration by the higher Calculus* is this:—
Let s be the solidity of a variable segment of the cone or pyramid, cut off by a plane parallel to the base; and let x be the height of that segment, and y its base; H being the height of the whole cone or pyramid, and A its base. Then

$$H^2 : x^2 :: A : y.$$

$$\therefore y \doteq \frac{x^2}{H^2} A, \text{ and } \partial s = y \partial x \doteq A \frac{x^2}{H^2} \partial x.$$

$$\therefore s \doteq \tfrac{1}{3} A \frac{x^3}{H^2} + (C = 0).$$

Let x become H, $: s$ becomes S.

$$\therefore S \doteq \tfrac{1}{3} A \times H.$$

Answers to unresolved exercises.

A. ·03316 − f. | B. 56·69 − c. in. | C. 8·482 +.

Exercise 1.

$$S \doteq \frac{3 \times 3 \times 8}{3} \times \Pi = 24\Pi = 75{\cdot}40-.$$

Exercise 2.

$$S \doteq \frac{450 \times 450 \times 315}{3 \times 27} = \frac{150 \times 150 \times 35}{1} = 787{,}500 \text{ c. yds.}$$

Exercise 3.

$$A \doteq \tfrac{5}{2} \times \tfrac{5}{2} \times 2{\cdot}5981 = 16{\cdot}2381.$$
$$S \doteq 16{\cdot}2381 \times 5 \div 3 = 27{\cdot}0635 \text{ c. ft.}$$
$$27{\cdot}0635 \times 170 \doteq 4601 - \text{lb.}$$

Exercise 4.

Diameter of base $\doteq 10{\cdot}5$. Radius $\doteq 5{\cdot}25$.
$$\therefore H \doteq \sqrt{(8{\cdot}75^2 - 5{\cdot}25^2)} = 7.$$
$$A \doteq 10{\cdot}5^2 \times \tfrac{1}{4}\Pi = 86{\cdot}590.$$

Exercise A.

$$S \doteq 1 \times {\cdot}07958 \times 1{\cdot}25 \div 3.$$

Exercise B.

$$A \doteq 2{\cdot}46^2 \times 4{\cdot}8284 = 29{\cdot}2195 + \text{sq. in.}$$

Exercise C.

Radius of base $\doteq \sqrt{(3{\cdot}9^2 - 3{\cdot}6^2)} = 1{\cdot}5.$

Problem III.

ANSWERS TO UNRESOLVED EXERCISES.

A. 8·232 + c. ft. | B. 2284·32.

Exercise 1.

$$4 + 3 + \sqrt{12} \doteq 10{\cdot}464 +.$$
$$10{\cdot}464 \times 1\tfrac{5}{4} \div 3 \doteq 6{\cdot}104.$$

Exercise A.

Area of base, $9 \times a$
..... of summit, $4 \times a$
$\sqrt{(9a \times 4a)} \doteq 6 \times a$
$19 \times a = 49{\cdot}364-.$

Exercise 2.

$\frac{9}{4} + \frac{25}{4} + \sqrt{(\frac{9}{4} \times \frac{25}{4})} \doteq 12\frac{1}{4} + 6\frac{1}{4} + 8\frac{3}{4} = 27.$
$27\frac{1}{4} \times 3 \div 3 \doteq 27\frac{1}{4}.$

Exercise 3.

$745^2 \ldots\ldots \doteq 555025$
$32^2 \ldots\ldots \doteq 1024$
$\sqrt{(745^2 \times 32^2)} = 745 \times 32 \doteq 23840$
579889
$\frac{1}{3}h \ldots\ldots \doteq 160$
$27 \begin{cases} (9)92782240 \text{ c. ft.} \\ (3)10309138- \\ 3436379 + \text{c. y.} \end{cases}$

$$\frac{92782240 \times 150}{28 \times 4 \times 20} \doteq \frac{2899445 \times 15}{7} = 6213096 \text{ tons.}$$

Exercise B.

By drawing a perp. from one corner of the square summit to the base we may find how much the base projects beyond the summit, viz. $\sqrt{(10{\cdot}6^2 - 9^2)} = 5{\cdot}6$. Therefore the side of the base exceeds that of the summit by 11·2, and the length of the former is 21·2. Then

Area of base $\doteq 21{\cdot}2^2 \ldots\ldots = 449{\cdot}44$
Area of summit $\doteq 10^2 \ldots\ldots = 100{\cdot}00$
$\sqrt{(21{\cdot}2^2 \times 10^2)} = 21 \times 10 \doteq 212{\cdot}00$
$761{\cdot}44.$

Exercise 4.

$E \doteq 201{\cdot}06+$. $\quad Ee \doteq 22739+$.
$e \doteq 113{\cdot}10-$. $\quad \sqrt{Ee} \doteq 150{\cdot}80-$.

Problem VII.

Exercise 1.

2 ft. 8 in.	1 ft. 9 in. × 1 ft. 4 in ≐ 2 ft. 4 pts.
2 „ 8 „	9 ft. 10 in. × $2\frac{1}{3}$ ≐ 22 ft. 11 p. 4 s.
4 „ 6 „	$S \doteq 22$ ft. 11 p. 4 s. ÷ 3
9 „ 10 „	= 3 ft. 9 p. 10 s. 8 in

Exercise A.

$\frac{1}{6}(2L + L') = \frac{1}{6}(3 \times 85{\cdot}4 + 12) = 44{\cdot}7$.
$H \times B = 26{\cdot}9 \times 26{\cdot}9 \times 2 = 1447{\cdot}22$.
$1447{\cdot}22 \times 44{\cdot}7 \doteq 64691-$, Ans.

Problem VIII.

Exercise 1.

$A \doteq 16 \times 7 \quad = 112$
$A' \doteq 10 \times 5 \quad = 50$
$4a \doteq 4(13 \times 6) = 312$
474
4
Ans. 1 ft. 168 c. in. = 1896 c. inches.

Exercise A.

	Ft.	in.	ft.	in.	sq. ft.	p.	s.
$A \doteq$	4	: 9 ×	3	: 3 =	15	: 5	: 3
$A' =$	5	: 3 ×	4	: 3 =	22	: 3	: 9
$4a =$	10	: 0 ×	7	: 6 =	75	: 0	: 0
					112	: 9	: 0
					9	: 4	: 9
112					6)122	: 1	: 9
					20	: 4	: $3\frac{1}{2}$ = 20·3576+ c. ft.

20·3576 c. ft. = ·754− c. yd. = ·942+ ton. Ans.

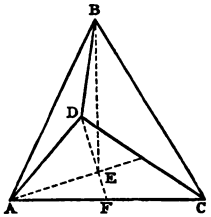
PROBLEM IX.

DEMONSTRATION OF THE RULE.

The numbers in the table are the solid contents of regular bodies, of the respective names, when the lineal side is 1. They are thus computed.

For the Tetrahedron.

Let ADC be any one of the four faces, regarded as a base. From the opposite angular point B, regarded as the vertex, suppose BE drawn perp. to the plane ADC. The point E will evidently be the centre of the base. Draw DE, AE; and produce DE to F: DF will be perp. to AC. Then, each of the lineal sides AD, AC, &c., being 1, $DF^2 \doteq AD^2 - AF^2 = 1^2 - (\frac{1}{2})^2 = \frac{3}{4}$; $DF \doteq \sqrt{\frac{3}{4}} = \frac{1}{2}\sqrt{3}$, and $\triangle ADC = \frac{1}{2}AC \times DF = \frac{1}{2} \times \frac{1}{2}\sqrt{3} = \frac{1}{4}\sqrt{3}$.



Again, $\angle EAF \doteq 30°$; $\therefore EF \doteq AE \sin 30° = AE \times \cdot 5 = \frac{1}{2}DE = \frac{1}{3}DF$. $\therefore DE = AE \doteq \frac{2}{3}DF$, and $AE^2 = \frac{4}{9}DF^2 = \frac{4}{9}$ of $\frac{3}{4} = \frac{1}{3}$. Also $BE^2 = AB^2 - AE^2 \doteq 1 - \frac{1}{3} = \frac{2}{3}$.

Now, pyramid $ABCD \doteq \frac{1}{3}\triangle ADC \times BE = \frac{1}{3} \times \frac{1}{4}\sqrt{3} \times \sqrt{\frac{2}{3}} = \frac{1}{12}\sqrt{2} = 0{\cdot}117851+$.

The tabular number *for the Hexahedron* needs no demonstration, since $1^3 \doteq 1$.

For the Octahedron.

The octahedron is composed of two square pyramids on the same base, that base being the square of the lineal side, and the united heights of the two pyramids being equal to the diagonal of the same square. All this will be evident by examining a model of the figure. $\therefore S \doteq \frac{1}{3} \times 1 \times \sqrt{2} = \frac{1}{3}\sqrt{2} = 0{\cdot}471405-$, when the lineal side is 1.

For the Dodecahedron and the Icosahedron, the investigation is more intricate, and as it extends to a greater length than can conveniently be given here, the reader is referred for it to the Key to the Author's *Complete Treatise*, Part *v. Ch. ix.*

The content of any one of the regular bodies being found whose lineal side is 1, that of a similar body whose side is L may be found by proportion; for, calling the number found for the former, N, we have by Pr. XLVII. of this Chapter, $1^3 : L^3 :: N : S$. $\therefore S \doteq L^3 \times N$.

Exercise A.

$(1{\cdot}2^3 = 1{\cdot}728) \times 2{\cdot}1817 \doteq 3{\cdot}770-$, Ans.

Problem XI.

Note to the Demonstration. Both rules may also be deduced from either of the rules for the following problem, by taking the case in which the segment becomes the half sphere, the height and the radius of the base being then each equal to the radius of the sphere. The content thus found, being doubled, gives Rule 1, from which Rule 2 follows.

A beautiful geometrical demonstration is given in the 21st Pr. of the 3d Book of the Supplement to Playfair's Geometry, along with the previous propositions. It is there shown that every sphere is two-thirds of the circumscribing cylinder. Consequently

$$S \doteq \tfrac{2}{3}D^2 \times \tfrac{1}{4}\Pi \times D = D^3 \times \tfrac{1}{6}\Pi.$$

Cor. The solid content of a hemisphere is two-thirds of that of a cylinder, and double that of a cone having the same base and height.

Answers to unresolved exercises.

A. 5277− millions of cubic miles.

B. 29·33+ lb. | C. 728·1 + lb.

Exercise 1.

$S \doteq (1{\cdot}75)^3 \times {\cdot}5236 = 5{\cdot}359375 \times {\cdot}5236.$

Or, $S \doteq (1\tfrac{3}{4})^3 \times {\cdot}5236 = \tfrac{343}{64} \times {\cdot}5236 = 2{\cdot}806+.$

Then, 1728 : 2·806 :: 1820 : 2·955+.

Exercise 2.

$S \doteq 44^3 \times {\cdot}5236 = 85184 \times {\cdot}5236 = 44602 +$ &c.

Exercise 3.

$882{,}000^3 \doteq 686{,}128{;}968{,}000{;}000{,}000 = C.$
$S.$ of Sun $\doteq C \times \cdot 5236 = 360{,}480{;}000{,}000{;}000{,}000.$
$S.$ of Earth $\doteq 7912^3 \times \cdot 5236 = 495{,}289{;}000{,}000 \times \cdot 5236 = 259{,}333{;}000{,}000.$
$360{,}480{;}000{,}000 \div 259{,}333 \doteq 1{;}390{,}000$ nearly.

The latter part may be done more simply thus :—

$$\frac{\text{Sun}}{\text{Earth}} \doteq \frac{882{,}000^3 \times \cdot 5236}{7912^3 \times \cdot 5236} = \frac{882\,000^3}{7912^3}.$$

Exercise A.

$2160^3 \doteq 10{,}077{;}696{,}000.$

Exercise B.

$S \doteq 216 \times \cdot 5236 = 113 \cdot 0976$ c. inches.
Weight $\doteq 469 \cdot 355 +$ oz. $= 29 \cdot 33 +$ lb.

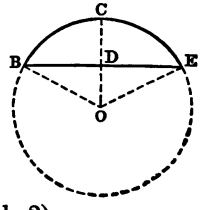
Exercise C.

$S \doteq 12\frac{5}{8} \times \cdot 5236 \times \frac{1}{2} = 4 \cdot 090625$ c. ft.

Problem XII.

Demonstration of the Rules.

Put the variable segment (BCE) $= s$, its height (CD) $= v$, and the radius (BD) of its base $= y$. Then



$$\begin{aligned}
\partial s &\doteq \Pi y^2 \partial v \\
&= \Pi(D - v)v\partial v \\
&= \Pi(Dv - v^2)\partial v. \\
\therefore s &\doteq \Pi(\tfrac{1}{2}D - \tfrac{1}{3}v)v^2 \\
&\quad (+ C = 0) \\
&= \tfrac{1}{6}\Pi(3\mathrm{D} - 2v)v^2 \text{ (which is Rule 2)} \\
&= \tfrac{1}{6}\Pi\{(3\mathrm{D} - 3v)v + v^2\} \times v \\
&= \tfrac{1}{6}\Pi(3y^2 + v^2) \times v \text{ (which is Rule 1).}
\end{aligned}$$

NOTE. Another demonstration might have been given by subtracting the content of the cone BOE from that of the spherical sector BCEO, the content of the latter being proved to be $= \frac{1}{3}$ surf. of seg. $\times$ radius of sphere, in the same manner as the rules in the last problem were demonstrated.

ANSWERS TO UNRESOLVED EXERCISES.

A. 2136+. | B. 2520(5−). | C. 2·736−.

EXERCISE 1.

By Rule 1,

$$S \doteq (3 \times 16 + 4) \times 2 \times \cdot 5236 = 104 \times \cdot 5236.$$

EXERCISE 2.

From the Cor. to Pr. LXX. of Ch. II.,

$$BD^2 \doteq AD \times DC = 2\tfrac{1}{4} \times 4 = 9 = r^2.$$
$$H^2 \doteq (2\tfrac{1}{4})^2 = 5\tfrac{1}{16}.$$

Then, by Rule 1,

$$S \doteq (27 + 5\tfrac{1}{16}) \times 2\tfrac{1}{4} \times \cdot 5236.$$

EXERCISE 3.

Here we first find OD = 11·9, then DC = 5. Then, by Rule 1,

$$S \doteq (3 \times 144 + 25) \times 5 \times \cdot 5236 = 1196 \cdot 4+.$$

EXERCISE A.

$$S \doteq 543 \cdot 9375 \times 7 \cdot 5 \times \cdot 5236.$$

EXERCISE B.

By Rule 2,

$$S \doteq 159 \times 302 \cdot 76 \times \cdot 5236.$$

Exercise C.

We first find $H = 4{\cdot}45 - 4 = 0{\cdot}45$. Then, by Rule 2,

$$S \doteq 25{\cdot}8 \times {\cdot}2025 \times {\cdot}5236.$$

Problem XIII.

DEMONSTRATION OF THE RULE.

Put KE $= H$, EF $= h$, KF $= H + h$, AE $= r$, and DF $= r$.
Then (by Pr. XII., Rule 1),

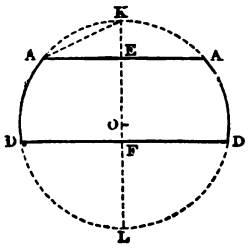
Seg. AKA $\doteq (3r^2H + H^3) \times \frac{1}{6}\Pi$, and

Seg. DKD $\doteq \{3r^2(H + h) + (H + h)^3\} \times \frac{1}{6}\Pi$.

$\therefore$ Zone AADD $\doteq$

$$(3r^2H - 3r^2H + 3r^2h + 3H^2h + 3Hh^2 + h^3) \times \tfrac{1}{6}\Pi.$$

But, DK being supposed drawn,



$AK^2 \doteq KE \times KL$; and

$DK^2 \doteq KF \times KL$.

$$\therefore \frac{AK^2}{KE} \doteq KL = \frac{DK^2}{KF},$$

or $AK^2 \times KF \doteq DK^2 \times KE$.

That is, $(r^2 + H^2) \times (H + h) \doteq \{r^2 + (h + H)^2\} \times H$;

or, $r^2H + r^2h + H^3 + H^2h \doteq r^2H + Hh^2 + 2H^2h + H^3$;

or, $r^2h \doteq r^2H - r^2H + H^2h + Hh^2$,

or, $3r^2h \doteq 3r^2H - 3r^2H + 3H^2h + 3Hh^2$.

If we now substitute $3r^2h$, for its value just found, in the above expression for the zone, we have

$$\text{Zone AADD} = (3r^2h + 3r^2h + h^3) \times \tfrac{1}{6}\Pi.$$

ANSWERS TO UNRESOLVED EXERCISES.

A. 14·89+. | B. 384·0 + c. feet.

Exercise 1.

$$S \doteq \{3(20^2 + 15^2) + 20^2\} \times 20 \times {\cdot}5236.$$
$$= 39500 \times {\cdot}5236 = 23824-.$$

Exercise 2.

$$\begin{array}{l|l} OE \doteq \sqrt{(65^2 - 52^2)} = 39 & r^2 \doteq 2704 \\ OF \doteq \sqrt{(65^2 - 60^2)} = 25 & r'^2 \doteq 3600 \\ OE - OF = h \ldots\ldots \doteq 14 & r^2 + r'^2 \doteq 6304 \end{array}$$

$$(3 \times 6304 + 196) \times 14 \doteq 267512.$$

Exercise A.

$$S \doteq (16{\cdot}71 + 2{\cdot}25) \times 1{\cdot}5 \times {\cdot}5236.$$

Exercise B.

We first find the squares of the radii of the two ends to be 2704 and 3969 sq. inches, the height, h, being 55 inches. Then

$$S \doteq 23044 \times 55 \times {\cdot}5236 = 663621 + \text{c. in.}$$

Problem XVII.

Answers to unresolved exercises.

A. 43·17 −. | B. 0·2500 +.

Exercise 1.

Length of ring $\doteq 8 \times 3{\cdot}1416 = 25{\cdot}1328.$
A. of cross sec. $\doteq 2{\cdot}375^2 \times {\cdot}7854 = 4{\cdot}430147.$
$25{\cdot}1328 \times 4{\cdot}430147 \doteq 111{\cdot}3 +.$

Exercise 3.

Length of ring $\doteq 4{\cdot}06 \times 3{\cdot}1416 = 12{\cdot}755 -.$
A. of cross sec. $\doteq {\cdot}21^2 \times {\cdot}7854 = {\cdot}034636\,14.$
$12{\cdot}755 \times {\cdot}034636 \doteq {\cdot}4418 -.$

EXERCISE A.

Length of ring $\doteq 5{\cdot}4 \times 3{\cdot}1416 = 16{\cdot}96464$.
A. of cross sec. $\doteq 1{\cdot}8^2 \times {\cdot}7854 = 2{\cdot}544696$.
$16{\cdot}96464 \times 2{\cdot}544696 \doteq 43{\cdot}17-$.

EXERCISE B.

Length of ring $\doteq 1 \times 3{\cdot}1416 = 3{\cdot}1416$.
A. of cross sector $\doteq 1 \times {\cdot}07958 = {\cdot}07958$.
$3{\cdot}1416 \times {\cdot}07958 \doteq {\cdot}2500+$.

Or more simply thus:—

$$S \doteq 1 \times \Pi \times \frac{1}{4\Pi} = \tfrac{1}{4}.$$

PROBLEM XVIII.

DEMONSTRATION OF THE RULE.

Let AC, in the following diagram, represent the fixed axis, F, whether transverse or conjugate; and BD, the revolving axis, R.

Put s for the variable segment (LCL′) of the spheroid; v for its abscissa (CQ); and y for its ordinate (LQ). Then, by a familiar property of the ellipse,

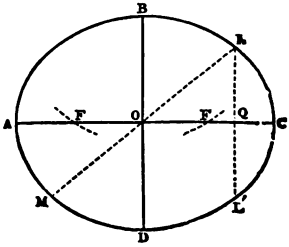
$$F^2 : R^2 :: \mathrm{AQ} \times \mathrm{QC} = (F-v)v : \mathrm{LQ}^2 = y^2.$$

$$\therefore y^2 \doteq \left(\frac{R}{F}\right)^2 \times (Fv - v^2).$$

$$\therefore \partial s = \Pi y^2 \partial v$$
$$\doteq \Pi\left(\frac{R}{F}\right)^2 \times (Fv - v^2)\partial v.$$

$$\therefore s \doteq \Pi\left(\frac{R}{F}\right)^2 \times (\tfrac{1}{2}Fv^2 - \tfrac{1}{3}v^3),$$

the constant being $= 0$.



Now let v become F, s becoming S. Then

$$S \doteq \Pi\left(\frac{R}{F}\right)^2 \times (\tfrac{1}{2}F^3 - \tfrac{1}{3}F^3) = \tfrac{1}{6}\Pi \times R^2.F.$$

COR. 1. Hence it appears that a hemi-spheroid is equal to two-thirds of a cylinder, or to the double of a cone, on the same base and of the same height.

COR. 2. The prolate spheroid is to the oblate, on the same axes, as the conjugate axis to the transverse.

ANSWERS TO UNRESOLVED EXERCISES.

A. 18·85−. | B. 25·13+.

EXERCISE 1.

Since the spheroid is oblate, the fixed axis is the shorter. Therefore

$$R \doteq 3\,;\ F \doteq 2\,;\ R^2 \times F \times \cdot 5236 \doteq 18 \times \cdot 5236.$$

EXERCISE 2.

This being a prolate spheroid, the fixed axis is the longer, and therefore

$$R \doteq 2\,;\ F \doteq 3\,;\ R^2 \times F \times \cdot 5236 \doteq 12 \times \cdot 5236.$$

EXERCISE 3.

$$R \doteq 7925{\cdot}5\,;\ F \doteq 7899\,;\ R^2 \doteq 62{;}813{,}550.$$
$$R^2 \times F \doteq 496{,}164{;}23(1{,}000).$$

EXERCISE A.

$$R \doteq 3\,;\ F \doteq 4\,;\ R^2 \times F \doteq 36.$$

EXERCISE B.

$$R \doteq 4\,;\ F \doteq 3\,;\ R^2 \times F \doteq 48.$$

Problem XLIII.

Exercise 1.

Mean dimensions, 49 ft. 6 in., 19 ft. 8 in., 10 ft. 10 in.
Solid content, $10546\frac{1}{4}$ c. feet.

Exercise A.

Mean dimensions, 27 ft. 9 in., 9 ft. 1 in., 3 ft. $2\frac{1}{4}$ in.
The solid content is 803 c. ft. 5 p. 5 − s., or 803·45 − c. feet = 29·7574 + c. yards.
The payment is £2·7377 − .

Problem XLVII.

Answers to unresolved exercises.

A. 157·464.
B. 30·20 + cwt.
C. 49·16 − times.
D. 60 st. 13 − lb., and 36 st. 8 − lb.
E. 4 lb. and $9\frac{1}{8}$ lb.

Exercise 1.

$20^3 = 8000 : 15^3 = 3375 :: 6480 : 2734 -$.

Exercise 2.

$5^3 = 125 : 6^3 = 216 :: 10$ st. : 17 st. 3·92 lb.

Exercise 3.

$1^3 : 8^3 = 512 :: 72$ lb. : 36864 lb. = 16 t. 8 c. 16 lb.

Exercise A.

$(\frac{2}{4})^3 : (1\frac{3}{4}{}^5)^3$, or $25^3 : 135^3$, or $5^3 : 27^3 :: 1 : 157{\cdot}464$.

Exercise B.

$6^3 : (5\frac{1}{2})^3$, or $(\frac{1}{3}^6)^3 : (\frac{1}{3}^6)^3$, or $18^3 : 16^3$,
or $9^3 : 8^3$, or $729 : 512 :: 43$ cwt. : 30·20+cwt.

Exercise C.

$2160^3 : 7912{\cdot}5^3$, or $17280^3 : 63300^3$, or
$864^3 : 3165^3 :: 1 : 49{\cdot}16-$.

Exercise D.

$(6\frac{1}{2})^3 : (5\frac{3}{4})^3$, or $(\frac{2}{4}^6)^3 : (\frac{2}{4}^3)^3$, or $26^3 : 23^3$, or
17576 : 12167 :: 88 st. : 60 st. 13 – lb.
And 17576 : 12167 :: 739 lb. : 512 – lb.

Exercise E.

To find the weight of the upper portion, $15^3 : 10^3$, or
$3^3 : 2^3$, or 27 : 8 :: 13·5 : 4.

CHAPTER VI.

PROMISCUOUS EXERCISES.

ANSWERS TO UNRESOLVED EXERCISES.

A. 13·84 | B. 2 ft. 11 in. | C. 36·50 – ft.

D. No error at all worth estimating, being less than ·006 of a square chain.

E. Almost exactly 30 tons. | F. 1545+times.

G. 22·30+ and 75·74+, or 2·08+ and 55·52+.

H. 11·8125 sq. feet. | K. 3·534+.

L. 103¼ sq. in., nearly, or ·51626 of the whole.

Exercise 1.

$3^2 + 4^2 + 12^2 \doteq 169 = 13^2$.

Exercise A.

The diagonals of a rhombus bisect each other at right angles. The rhombus is therefore divided into four $\triangle$s, in any one of which we have given the diagonal 8·65, and one leg 5·19, from which we find the other leg 6·92. The double of this is the longer diagonal.

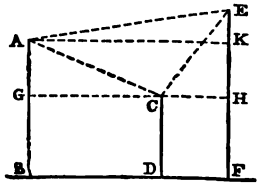
Exercise B.

In every case of reflection by a plane mirror the reflected figure appears of the same size as the real figure, and as much behind the mirror as the real object is before it. The mirror, therefore, being placed vertically, is half way between the eye of the spectator and his reflected figure; and, consequently, if st. lines be supposed to extend from his eye to the highest and lowest points of the reflected figure, they will intercept a length upon the mirror equal to half the height of the reflected figure, which is the same as that of the real figure. The looking-glass must therefore be half the height of the person using it, that he may see himself from head to foot.

Exercise C.

AB, CD, and EF being the three poles and the other lines being drawn as in the diagram, we have, in $\triangle$ EHC,

$CE = 20$, $CH = 12$,
and $\therefore EH = 16$.
$\therefore HF \doteq 15 = BG$,
and $AG \doteq 25 - 15 = 10$.
Hence, in $\triangle$ AGC,
$AC \doteq 26$, $AG \doteq 10$,
and $\therefore GC \doteq 24$.



Lastly, in $\triangle$ AKE,

$$AK \doteq GH = GC + CH = 24 + 12 = 36,$$
$$\text{and } KE \doteq EF - AB = 31 - 25 = 6.$$
$$\therefore AE \doteq \sqrt{1332} = 36{\cdot}50 -.$$

Exercise 2.

$$(\tfrac{1}{4})^3 : 3^3 :: 1 : 1728.$$
$$\text{Or, } 3^3 . \frac{\Pi}{6} \div \left\{\left(\frac{1}{4}\right)^3 . \frac{\Pi}{6}\right\} \doteq 3^3 \div \left(\frac{1}{4}\right)^3 = 1728.$$

Exercise 3.

$$\text{Piece wasted} \doteq 10^2 - 10^2 \times {\cdot}7854 - = 100 - 78{\cdot}54 -.$$

Exercise 4.

The large square would be divided into four squares, each 5 inches in lineal dimensions; and out of each of these a circle would be cut of 5 inches diameter. The part wasted in each small square would be $25 \times (1 - {\cdot}7854) = 25 \times {\cdot}2146$; and, in all the four pieces, $100 \times {\cdot}2146 = 21{\cdot}46$.

Exercise 5.

By the reverse formula of Pr. II. of Ch. IV.,

$$\text{Diagonal} \doteq \sqrt{2A} = \sqrt{36} = 6.$$

Exercise 6.

The surface of the cube consists of six squares, every side of which is the same as one edge of the cube, viz. $\sqrt[3]{125}$, or 5.

$$\therefore \text{Surface} \doteq 6 \times 5^2 = 150.$$

EXERCISE D.

Expressing the dimensions in chains, we have

$$\text{Area of 1st } \Delta = \sqrt{(12{\cdot}685 \times 7{\cdot}685 \times 4{\cdot}685 \times {\cdot}315)}$$
$$= \sqrt{(143{\cdot}865-)} \quad = 11{\cdot}994+, \text{ and}$$
$$\text{Area of 2d } \Delta = \sqrt{(9 \times 4 \times 4 \times 1)} = 12{\cdot}000.$$

The reason of the slight difference of area caused by so great an error in the measurement of one side, is, that, if we take 800 as the base and construct the two triangles upon that base, the line joining their vertices comes to be very nearly parallel to the base: and, consequently, the two triangles are very nearly equal.

Or, if on a base of 800 we construct an isosceles triangle having each of its equal sides 500,—if we then draw through the vertex a line parallel to the base, and if, with either extremity of the base as a centre and a radius = 500, we describe a circle, we find that it cuts the parallel line in two points, the first of which is, of course, 500 links distant from the other extremity of the base, and the second is as nearly as possible 1237.

EXERCISE E.

The area cleared was = 10 yds. × 28 in. = 10 × 3 × 12 × 28 in. Then

As $10 \times 3 \times 12 \times 28$: $4840 \times 9 \times 144$ :: 108, or

As 7 : $484 \times 3 \times 3$ :: 108 : weight in pounds.

$$\therefore \text{Weight in tons} = \frac{484 \times 3 \times 3 \times 108}{7 \times 112 \times 20} = \frac{121 \times 3 \times 3 \times 27}{7 \times 28 \times 5}$$

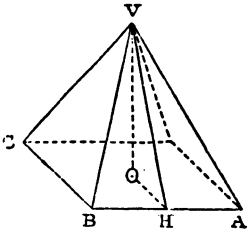
= 30 almost exactly.

EXERCISE F.

By Pr. XX. of Ch. IV., the conv. surf. of the roller is 4058·9+ sq. inches. By this we divide 6,272,640, being *the number* of sq. inches in an acre.

EXERCISE 7.

Each side of the pyramid is a triangle, as BVA, whose altitude is VH. To find VH we have recourse to the triangle VOH, which is right-angled at O. In this, $HO = \frac{1}{2}BC \doteq 17{\cdot}5$, and $VO \doteq 42$. $\therefore HV \doteq 45{\cdot}5$ (by Pr. I. of Ch. III.).



$\therefore$ Area of $\Delta BVA \doteq \frac{1}{2}$ of $45{\cdot}5 \times 35 = 796{\cdot}25$

4

Area of four sides,.....3185
Area of base,.....1225

Ans. 4410.

EXERCISE 8.

Area of 3 sides $\doteq 3\sqrt{(3{\cdot}405 \times 1{\cdot}955 \times {\cdot}725 \times 725)}$
$= 3\sqrt{3{\cdot}4989} = 5{\cdot}6116.$
Area of base $\doteq 1{\cdot}45^2 \times {\cdot}43301 = 0{\cdot}9104.$

EXERCISE G.

The areas of the two circles and their segments (found by Rule 6 of Pr. XVI. of Ch. IV.) are as follows:—

Of greater circle.	Of smaller circle.
Circle.....81·713	Circle,.....28·274
Smaller seg. 1·944	Smaller seg. 4·027
Greater seg. 79·769	Greater seg. 24·247

Since the two circles may have two different positions, relatively to each other, there arise two values to each result, viz. $24{\cdot}247 - 1{\cdot}944$, $79{\cdot}769 - 4{\cdot}027$, $4{\cdot}027 - 1{\cdot}944$, and $79{\cdot}769 - 24{\cdot}247$.

EXERCISE H.

Surf. of 6 sides of pyr. $\doteq 1{\cdot}5 \times 3 \times 6 = 27.$
16 : 9 :: 27 : $15{\cdot}1875$ = surf. of pyr. cut off.
$27 - 15{\cdot}1875 \doteq$ surf. of 6 sides of frustum.

Exercise K.

Area of outer ellipse $\doteq 21 \times \cdot 7854$
..... inner..... $\doteq 16\frac{1}{2} \times \cdot 7854$
..... ring $\doteq 4\frac{1}{2} \times \cdot 7854 = 3 \cdot 534+$.

Exercise L.

Circumf. of base of cone $\doteq$	18·8496
Convex surf. of cone..... =	94·248.
Overlapping surface..... =	2·5
	96·748
Surf. of whole plate..... =	200
Portion wasted.....	103·252.

FINIS.